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PERIODIC q -DIFFERENCE EQUATIONS

by

ROTCHANA CHIEOCHAN

A DISSERTATION

Presented to the Faculty of the Graduate School of the
MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

2012

Approved by:

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DEDICATION

This dissertation is dedicated in loving memory to

Rungroj Chieochan

February 12, 1976 – December 25, 2002

PUBLICATION DISSERTATION OPTION

This dissertation consists of the following five articles as the following:

I. FLOQUET THEORY FOR q -DIFFERENCE EQUATIONS, pages 22–39,

II. THE BEVERTON–HOLT q -DIFFERENCE EQUATION, pages 40–53,

III. STABILITY FOR HAMILTONIAN q -DIFFERENCE SYSTEMS, pages 54–77,

IV. EXISTENCE OF PERIODIC SOLUTIONS OF A q -DIFFERENCE BOUNDARY VALUE PROBLEM, pages 78–94, and

V. POSITIVE PERIODIC SOLUTIONS OF HIGHER-ORDER FUNCTIONAL q -DIFFERENCE EQUATIONS, pages 95–109,

which are intended for submission in

1. ADVANCES IN DIFFERENCE EQUATIONS AND APPLICATIONS,
2. JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS,
3. INTERNATIONAL JOURNAL OF DIFFERENCE EQUATIONS,
4. ABSTRACT AND APPLIED ANALYSIS, and
5. SELCUK JOURNAL OF MATHEMATICS.

ABSTRACT

The concept of periodic functions defined on the real numbers or on the integers is a classical topic and has been studied intensively, yielding numerous applications in every kind of science. It is of importance that the real numbers and the integers are closed with respect to addition. However, for a number $q > 1$, the so-called q -time scale, i.e., the set of nonnegative integer powers of q , is not closed with respect to addition, and therefore it was not possible to define periodic functions on the q -time scale in an obvious way. In this thesis, this important open problem has been resolved and the definition of periodic functions defined on the q -time scale is given. Using this new definition of periodic functions defined on the q -time scale, five distinct results involving periodic solutions of various kinds of q -difference equations are presented, namely as follows. First, Floquet theory for q -difference equations is established. Second, the Cushing–Henson conjecture is proved for periodic solutions of the Beverton–Holt q -difference equation, resulting in applications in the study of biology, in particular population models. Third, stability for Hamiltonian q -difference systems is investigated. Fourth, the existence of periodic solutions of a q -difference boundary value problem is examined by applying the well-known Mountain Pass theorem. Fifth, the existence of positive periodic solutions of higher-order functional q -difference equations is studied by applying the well-known fixed-point theorem in a cone. Besides these five research papers that are based on the newly introduced definition of periodic functions on the q -time scale, this thesis also contains an introduction, a section on time scales calculus, a section on quantum calculus, and a conclusion.

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NOMENCLATURE

\mathbb{T}	A Time Scale
\mathbb{R}	Set of Real Numbers
\mathbb{N}	Set of Natural Numbers
\mathbb{N}_0	$\mathbb{N} \cup \{0\}$
\mathbb{N}_0^2	The Set $\{0, 1, 4, 9, 16, \dots\}$
$h\mathbb{Z}$	The Set $\{\dots, -2, -1, 0, 1, 2, \dots\}$
\mathbb{C}	Set of Complex Numbers
$\overline{q^{\mathbb{Z}}}$	The Set $\{\dots, q^{-2}, q^{-1}, 1, q, q^2, \dots\}$ for $q > 1$.
$q^{\mathbb{N}_0}$	The Set $\{1, q, q^2, q^3, \dots\}$ for $q > 1$.
σ	Forward Jump Operator
ρ	Backward Jump Operator
μ	Graininess Function
Δ	Delta Derivative Operator
Δ	Forward Difference Operator
\mathcal{R}	Set of Regressive Functions
$A > 0$	A Positive Definite Matrix
$A \geq 0$	A Positive Semi-definite Matrix
\oplus	Addition on Time Scales
\ominus	Subtraction on Time Scales
\odot	Multiplication on Time Scales
$e_A(\cdot, \cdot)$	Matrix Exponential on Time Scales

1. INTRODUCTION

Differential equations began with Leibniz, the Bernoulli brothers, and others from the 1680s, not long after Newton's fluxional equations in the 1670s. In 1676, English physicist Isaac Newton solved his first differential equation and was working with what he called fluxional equations.

In 1693, German mathematician Gottfried Leibniz solved his first differential equation and that same year Newton published the results of previous differential equation solution methods a year that is said to mark the inception for the differential equations as a distinct field in mathematics.

Swiss mathematicians, brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748), in Basel, Switzerland, were among the first interpreters of Leibniz' version of differential calculus. They were both critical of Newton's theories and maintained that Newton's theory of fluxions was plagiarized from Leibniz' original theories, and went to great lengths, using differential calculus, to disprove Newton's Principia, on account that the brothers could not accept the theory, which Newton had proven, that the earth and the planets rotate around the sun in elliptical orbits. The first book on the subject of differential equations, supposedly, was Italian mathematician Gabriele Manfredi's 1707 *On the Construction of First-degree Differential Equations*, written between 1701 and 1704, published in Latin. The book was largely based on the views of the Leibniz and the Bernoulli brothers. Most of the publications on differential equations and partial differential equations, in the years to follow, in the 18th century, seemed to expand on the version developed by Leibniz, a methodology, employed by those as Leonhard Euler, Daniel Bernoulli, Joseph Lagrange, and Pierre Laplace.

For the recent era, there are many mathematicians who have studied and developed the theory of differential and also difference equations as found in general. The study about the periodic solutions of differential and difference equations is the one significant topic in which we are familiar with. The search for periodic solutions and the examination of their behavior are of interest not only from the purely mathematical point of view but

also because the periodic regimes of real physical systems usually correspond to periodic solutions in the mathematical description of these systems (see Auto-oscillation; Forced oscillations; Oscillations, theory of; Non-linear oscillations; Relaxation oscillation). However, this is a very difficult problem, since there are no general methods for establishing whether periodic solutions exist for a particular system. Various arguments and methods are used in different cases. Many of them relate to perturbation theory, e.g., the harmonic balance method and the Krylov-Bogolyubov method of averaging or the method of the small parameter, and they also touch upon research on bifurcation. Others relate to the qualitative theory of differential equations.

Differential or difference equations involving periodic functions play also an important role in many applications which are called Floquet equations and the study of Floquet equations is called Floquet theory. Although it is not necessary that a Floquet system has a periodic solution, it is possible to characterize all the solutions of such system and to give conditions under which a periodic solution does exist. In this work, we have also presented Floquet theory for the q -difference equations. However, the periodicity of functions defined on the q -time scale is unlike that on \mathbb{R} and \mathbb{Z} time scales because for any $s, t \in q^{\mathbb{N}_0}$, $s + t$ is not necessarily in $q^{\mathbb{N}_0}$. In other words, $q^{\mathbb{N}_0}$ is not closed under plus operation. In this work, we define periodic functions on q -time scale that will be introduced later. Furthermore, the geometrical interpreting of periodic functions is also considered by calculating the integral of the periodic functions over some intervals and then the value of those integrals becomes constants. Through this work we count on the periodicity idea for the q -time scale to develop the five articles shown in the contents. As already mentioned, the Floquet theory is the one of five articles included in this work and it will be described in more detail later. For the second article, we investigate the existence of the periodic solutions of the Beverton–Holt equations on the q -time scale and present a couple of Cushing–Henson conjectures which are analogue version of the difference or differential equations. As well known the Beverton–Holt model is a classic population model which has the applications for the population growth and delay. We continue the study of the periodic solutions in the fourth and fifth articles, but for the fourth we concentrate on the q -difference boundary value problem by applying the Mountain Pass Theorem while

in the fifth one we look over the higher-order functional q -difference equations and seek the positive periodic solutions for that given q -difference equations. In the third paper, the stability theorems for the Hamiltonian q -difference equations are presented which are developed from the continuous and discrete versions. The research of the stability of the difference (differential) Hamiltonian equations has been studied by many authors, e.g., Răsvan [27] and [17] (Krein and Jakubovič [22]).

Throughout this work, we have presented many significant results, theorems, and useful approaches dealing with the periodic solutions of the various q -difference equations which are developed parallel to some portions which are related to the periodic solutions of differential (difference) equations.

2. INTRODUCTION TO TIME SCALES

The study of dynamic equations on time scales unifies both continuous and discrete mathematical analysis. As a result, one can generalize a process to account for both cases, or any combination of the two. Since its inception, this area of mathematics has gained a great deal of international attention. Researchers have found applications of time scales to include heat transfer, population dynamics, and economics. In further sections, our results will be extended toward applications found in electrical engineering. For a more in-depth study of time scales, see Bohner and Peterson's books [5, 6].

2.1. BASIC DEFINITIONS

In this subsection, the basic results on time scales are introduced to be used in later sections.

Definition 2.1. A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers.

This is some common examples of time scales.

Example 2.2. Some common time scales include

- a. $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$;
- b. $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$ for $h > 0$;
- c. $\mathbb{T} = q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$ for $q > 1$;
- d. $\mathbb{T} = 2^{\mathbb{Z}} := \{2^k : k \in \mathbb{Z}\}$;
- e. the so-called harmonic numbers, $\left\{ H_n = \sum_{k=1}^n \frac{1}{k} : n \in \mathbb{N}_0 \right\}$;
- f. $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$;
- g. the Cantor set.

Any time scale that is a combination of any of the above sets is called a hybrid time scale. On the contrary, sets such as (a, b) and \mathbb{C} are not time scales.

Next, both the forward and the backward jump operators must be defined.

Definition 2.3. For $t \in \mathbb{T}$, the following statements are defined:

- a. The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is given by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}. \quad (2.1)$$

- b. The backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined as

$$\rho(t) := \sup \{s \in \mathbb{T} : s < t\}. \quad (2.2)$$

Definition 2.4. For any function $f : \mathbb{T} \rightarrow \mathbb{R}$, the function $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ is defined as

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}, \quad (2.3)$$

i.e., $f^\sigma = f \circ \sigma$.

Remark 2.5. Points in \mathbb{T} are classified as follows. If $\sigma(t) > t$, then t is said to be right-scattered. Similarly, if $\rho(t) < t$, then t is said to be left-scattered. If a point is both left-scattered and right-scattered, then it is said to be isolated. On the contrary, if $\sigma(t) = t$, then t is said to be right-dense. Similarly, if $\rho(t) = t$, then t is said to be left-dense. If a point is both left-dense and right-dense, then it is said to be dense. Table 2.1 gives a classification of points.

Definition 2.6. If \mathbb{T} is a time scale with a left-scattered maximum m , then the set $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.7. The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t. \quad (2.4)$$

Both the forward and the backward jump operators as well as the graininess function for some common time scales are given in Table 2.2.

Table 2.1. Classification of Points

$t < \sigma(t)$	t is right-scattered
$\rho(t) < t$	t is left-scattered
$\rho(t) < t < \sigma(t)$	t is isolated
$\sigma(t) = t$	t is right-dense
$\rho(t) = t$	t is left-dense
$\rho(t) = t = \sigma(t)$	t is dense

Table 2.2. Examples of Times Scales

\mathbb{T}	$\mu(t)$	$\sigma(t)$	$\rho(t)$
\mathbb{R}	0	t	t
\mathbb{Z}	1	$t + 1$	$t - 1$
$h\mathbb{Z}$	h	$t + h$	$t - h$
$q^{\mathbb{N}}$	$(q - 1)t$	qt	$\frac{t}{q}$
$2^{\mathbb{N}}$	t	$2t$	$\frac{t}{2}$
\mathbb{N}_0^2	$1 + 2\sqrt{t}$	$(\sqrt{t} + 1)^2$	$(\sqrt{t} - 1)^2$
H_n	$\frac{1}{n+1}$	H_{n+1}	H_{n-1}

2.2. DIFFERENTIATION

The delta (or Hilger) derivative is defined in the following. Then some useful properties dealing with the delta derivative are presented, Bohner and Peterson [5, 6].

Definition 2.8. Let $f : \mathbb{T} \rightarrow \mathbb{R}$. The delta derivative $f^\Delta(t)$ is the number (when it exists) such that given any $\varepsilon > 0$, there is a neighborhood U of t such that,

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

Properties of the delta derivative are considered in the next two theorems.

Theorem 2.9 (See [5, Theorem 1.16]). *Suppose $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function; let $t \in \mathbb{T}^\kappa$. Then the following results are produced.*

- a. *If f is differentiable at a point t , then f is continuous at t .*
- b. *If f is continuous at t , where t is right-scattered, then f is differentiable at t and*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \quad (2.5)$$

- c. *If f is differentiable at t , where t is right-dense, then*

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}. \quad (2.6)$$

- d. *If f is differentiable at t , then*

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t). \quad (2.7)$$

Remark 2.10. Note the following examples.

- a. When $\mathbb{T} = \mathbb{R}$, then (if the limit exists)

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

- b. When $\mathbb{T} = \mathbb{Z}$, then

$$f^\Delta(t) = f(t+1) - f(t) = \Delta f(t).$$

c. When $\mathbb{T} = q^{\mathbb{Z}}$ for $q > 1$, then

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

Next the linearity property as well as both the product and the quotient rules are considered.

Theorem 2.11 (See [5, Theorem 1.20]). *Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be differentiable at $t \in \mathbb{T}^{\kappa}$. Then the following results are produced.*

a. *For any constants α and β , the sum $(\alpha f + \beta g) : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with*

$$(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t). \quad (2.8)$$

b. *The product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t). \quad (2.9)$$

c. *If $g(t)g(\sigma(t)) \neq 0$, then the quotient $f/g : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \quad (2.10)$$

2.3. INTEGRATION

Now integrable functions on an arbitrary time scale will be considered. However, the following two concepts must first be introduced.

Definition 2.12. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be regulated if its left-sided and right-sided limits exist at all left-dense and right-dense points in \mathbb{T} , respectively.

Definition 2.13. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist at left-dense points in \mathbb{T} . The class

of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ is denoted by

$$C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, \mathbb{R}). \quad (2.11)$$

From the previous two definitions, we have the following theorem.

Theorem 2.14 (See [5, Theorem 1.60]). *Let $f : \mathbb{T} \rightarrow \mathbb{R}$.*

- a. *If f is continuous, then it is also rd-continuous.*
- b. *If f is rd-continuous, then it is also regulated.*
- c. *The jump operator σ is rd-continuous.*
- d. *If f is regulated or rd-continuous, then so is f^σ .*
- e. *Assume f is continuous. If $g : \mathbb{T} \rightarrow \mathbb{R}$ is regulated or rd-continuous, so is $f \circ g$.*

Definition 2.15. A continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be pre-differentiable with (region of differentiation) D , provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of \mathbb{T} , and f is differentiable at each point $t \in D$.

Next, we consider when the existence of pre-antiderivatives is guaranteed.

Theorem 2.16 (See [5, Theorem 1.70]). *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a regulated function. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \quad \text{for all } t \in D.$$

Any such function F is called a pre-antiderivative of f .

Definition 2.17. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a regulated function and let F be a pre-antiderivative of f . Then the Cauchy integral of f is given by

$$\int_a^b f(t) \Delta t = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

Example 2.18. Let $a, b \in \mathbb{T}$ and f be rd-continuous. Note the following examples.

a. When $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t) \Delta t = \int_a^b f(t) dt.$$

b. When $[a, b]$ contains only isolated points, then

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{t \in [a, b)} \mu(t) f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ - \sum_{t \in [a, b)} \mu(t) f(t) & \text{if } a > b. \end{cases}$$

c. When $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ - \sum_{t=b}^{a-1} f(t) & \text{if } a > b. \end{cases}$$

c. When $\mathbb{T} = h\mathbb{Z}$, then

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{k=a/h}^{b/h-1} hf(hk) & \text{if } a < b \\ 0 & \text{if } a = b \\ - \sum_{k=b/h}^{a/h-1} hf(hk) & \text{if } a > b. \end{cases}$$

In the following theorem, the basic properties of integration are considered on time scales.

Theorem 2.19 (See [5, Theorem 1.77]). *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{\text{rd}}$, then*

a. $\int_a^b [f(t) + g(t)] \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t;$

b. $\int_a^b (\alpha f)(t) \Delta t = \alpha \int_a^b f(t) \Delta t;$

c. $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t;$

- d. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$
- e. $\int_a^b [f^\sigma(t)g^\Delta(t)]\Delta t = (fg)(b) - (fg)(a) + \int_a^b f^\Delta(t)g(t)\Delta t;$
- f. $\int_a^b [f(t)g^\Delta(t)]\Delta t = (fg)(b) - (fg)(a) + \int_a^b f^\Delta(t)g^\sigma(t)\Delta t;$
- g. $\int_a^a f(t)\Delta t = 0.$

Finally, a generalized form of the Leibniz rule is considered.

Theorem 2.20 (See [5, Theorem 1.117]). *Let $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$, and assume $f : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is continuous at (t, t) , where $t \in \mathbb{T}^\kappa$ with $t > a$. Additionally, assume that $f^\Delta(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$, there exists a neighborhood U of t independent of $\tau \in [a, \sigma(t)]$ such that*

$$|f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

- a. $g(t) := \int_a^t f(t, \tau)\Delta\tau$ implies $g^\Delta(t) = \int_a^t f^\Delta(t, \tau)\Delta\tau + f(\sigma(t), t);$
- b. $h(t) := \int_t^b f(t, \tau)\Delta\tau$ implies $h^\Delta(t) = \int_t^b f^\Delta(t, \tau)\Delta\tau - f(\sigma(t), t).$

2.4. EXPONENTIAL FUNCTIONS

Exponential functions on time scales are introduced in this section. Regressive functions on time scales are offered first.

Definition 2.21. The function $p : \mathbb{T} \rightarrow \mathbb{R}$ is said to be regressive, provided that

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}^\kappa.$$

The set of all regressive and rd-continuous functions is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

Some special operations for regressive functions on time scales must also be considered. The next three definitions will be given to introduce several properties of the exponential function on time scales.

Definition 2.22. Let $p, q \in \mathcal{R}$. The “circle plus” addition \oplus is then defined by

$$(p \oplus q)(t) = p(t) + (1 + \mu(t)p(t))q(t) \quad \text{for all } t \in \mathbb{T}^\kappa. \quad (2.12)$$

Definition 2.23. Let $p, q \in \mathcal{R}$. The “circle minus” subtraction \ominus is then defined by

$$(p \ominus q)(t) = \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \quad \text{for all } t \in \mathbb{T}^\kappa. \quad (2.13)$$

Definition 2.24. Let $n \in \mathbb{N}$ and $p \in \mathcal{R}$. The “circle dot” multiplication \odot is denoted with

$$n \odot p = p \oplus p \oplus p \oplus \dots \oplus p,$$

where n terms exist on the right-hand side of the equation.

The notion of the Hilger complex plane is introduced.

Definition 2.25. For $h > 0$, the Hilger complex numbers are defined by

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$

When $h = 0$, let $\mathbb{C}_0 = \mathbb{C}$.

The exponential function is expressed in terms of what is known as the cylinder transformation whose range is the set \mathbb{Z}_h , defined as follows.

Definition 2.26. For $h > 0$, the strip is defined as

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h} \right\}.$$

When $h = 0$, let $\mathbb{Z}_0 := \mathbb{C}$.

Definition 2.27. For $h > 0$, the cylinder transformation $\xi_h : \mathbb{C}_h \rightarrow \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \frac{1}{h} \text{Log}(zh + 1),$$

where Log represents the principal logarithm function. For $h = 0$, $\xi_0(z) = z$ is defined for all $z \in \mathbb{C}$.

The generalized exponential function is given as follows.

Definition 2.28. If $p \in \mathcal{R}$, then the exponential function is defined by

$$e_p(t, s) = \exp \left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right) \quad \text{for } s, t \in \mathbb{T}, \quad (2.14)$$

where the cylinder transformation $\xi_h(z)$ is the same as in Definition 2.27.

Definition 2.29. If $p \in \mathcal{R}$, then the linear dynamic equation

$$y^\Delta(t) = p(t)y(t) \quad (2.15)$$

is called regressive.

Theorem 2.30 (See [5, Theorem 2.33]). *Suppose that equation (2.15) is regressive and fix $t_0 \in \mathbb{T}$. Then the solution to the initial value problem*

$$y^\Delta(t) = p(t)y(t), \quad y(t_0) = 1 \quad (2.16)$$

is given by $e_p(\cdot, t_0)$.

The following theorem addresses the uniqueness of the solution for (2.16).

Theorem 2.31 (See [5, Theorem 2.35]). *If (2.15) is regressive, then the only solution of (2.16) is given by $e_p(\cdot, t_0)$.*

Some properties of the exponential function are stated as the following.

Theorem 2.32 (See [5, Theorem 2.36] and [6, Theorem 2.44]). *If $p, q \in \mathcal{R}$, then*

- a. $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- b. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- c. $e_{\ominus p}(t, s) = \frac{1}{e_p(t, s)}$;
- d. $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;
- e. $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- f. $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;
- g. $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s)$;
- h. $\left(\frac{1}{e_p(\cdot, s)} \right)^\Delta = -\frac{p}{e_p^\sigma(\cdot, s)}$;

Theorem 2.33 (See [5, Theorem 2.39]). *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c, \cdot)]^\Delta = -p[e_p(c, \cdot)]^\sigma$$

and

$$\int_a^b p(\tau)e_p(c, \sigma(\tau))\Delta\tau = e_p(c, a) - e_p(c, b).$$

2.5. MATRIX EXPONENTIAL

Before introducing the matrix exponential, the notion of regressive matrices must first be considered.

Definition 2.34. Let A be an $m \times n$ matrix-valued function defined on \mathbb{T} . If every entry of A is rd-continuous on \mathbb{T} , then A is said to be rd-continuous on \mathbb{T} .

It should be noted that the class of rd-continuous matrix-valued functions is abbreviated by

$$C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, \mathbb{R}^{m \times n}). \quad (2.17)$$

Remark 2.35. Consider the linear system of dynamic equations:

$$x^\Delta(t) = A(t)x(t), \quad (2.18)$$

where A is an $n \times n$ matrix defined on \mathbb{T} . The vector-valued function $v : \mathbb{T} \rightarrow \mathbb{R}$ is said to be a solution of (2.18) provided that $v^\Delta(t) = A(t)v(t)$ holds for all $t \in \mathbb{T}^\kappa$. The following definition, however, is necessary to discuss this system subject to some initial condition.

Definition 2.36. Let A be an $n \times n$ matrix-valued function defined on \mathbb{T} . A is said to be regressive if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, where I is the identity matrix.

The class of all rd-continuous and regressive matrix-valued functions is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{m \times n}). \quad (2.19)$$

The system (2.18) is said to be regressive provided $A \in \mathcal{R}$. The existence and uniqueness theorems are offered as follows before considering the solution to an initial value problem for (2.18).

Theorem 2.37 (See [5, Theorem 5.8]). *Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function defined on \mathbb{T} . Suppose that $f : \mathbb{T} \rightarrow \mathbb{R}^n$, $t_0 \in \mathbb{T}$, and $x_0 \in \mathbb{R}$. Then the initial value problem*

$$x^\Delta(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0 \quad (2.20)$$

has a unique solution $x : \mathbb{T} \rightarrow \mathbb{R}^n$.

Next, two special operations will be introduced.

Definition 2.38. Let A and B be regressive $n \times n$ matrix-valued functions defined on \mathbb{T} . The “circle plus” addition \oplus is then defined by

$$(A \oplus B)(t) = A(t) + (I + \mu(t)A(t))B(t) \quad \text{for all } t \in \mathbb{T}^\kappa. \quad (2.21)$$

The additive inverse \ominus (read “circle minus”) is defined by

$$\begin{aligned} (\ominus A)(t) &= -[I + \mu(t)A(t)]^{-1}A(t) \\ &= -A(t)[I + \mu(t)A(t)]^{-1} \quad \text{for all } t \in \mathbb{T}^\kappa. \end{aligned} \quad (2.22)$$

Next, both the matrix exponential on the time scale \mathbb{T} and some of its properties will be considered.

Definition 2.39. Suppose A is regressive and rd-continuous. Then the unique $n \times n$ matrix-valued solution to the IVP

$$X^\Delta(t) = A(t)X(t), \quad X(t_0) = I$$

is called the matrix exponential function and is denoted by $e_A(\cdot, t_0)$.

Example 2.40. Assume that A is an $n \times n$ matrix.

a. If $\mathbb{T} = \mathbb{Z}$, then

$$e_A(t, t_0) = \begin{cases} \prod_{\tau=t_0}^{t-1} [I + A(\tau)] & \text{if } A \text{ is never } -I \\ (I + A)^{t-t_0} & \text{if } I + A \text{ is a constant and invertible.} \end{cases}$$

b. If $\mathbb{T} = \mathbb{R}$, then

$$e_A(t, t_0) = \begin{cases} \exp \left\{ \int_{t_0}^t A(\tau) d\tau \right\} & \text{if } A \text{ is continuous and} \\ & A(s)A(t) = A(t)A(s) \text{ for all } s, t \in \mathbb{T} \\ e^{A(t-t_0)} & \text{if } A(t) \text{ is constant.} \end{cases}$$

c. If $\mathbb{T} = h\mathbb{Z}$, then

$$e_A(t, t_0) = \begin{cases} \prod_{\tau=t_0}^{t/h-1} [I + hA(h\tau)] & \text{if } A \text{ is regressive} \\ (I + hA)^{\frac{t-t_0}{h}} & \text{if } I + hA \text{ is a constant and invertible.} \end{cases}$$

d. If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, then

$$e_A(t, 1) = \prod_{\tau \in \mathbb{T} \cap (0, t)} [I + (q - 1)\tau A(\tau)].$$

Theorem 2.41 (See [5, Theorem 5.21]). *Let $e_A(\cdot, t_0)$ be as in Definition 2.39. Then for $r, s, t \in \mathbb{T}$, the following results are derived:*

- a. $e_A(t, t) = e_0(t, s) \equiv I$.
- b. $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$.
- c. $e_A^{-1}(t, s) = e_A(s, t) = e_{\ominus A^T}^T(t, s)$.
- d. $e_A(t, s)e_A(s, r) = e_A(t, r)$.

Next the solution to linear systems will be found using a variation of parameters.

Theorem 2.42 (See [5, Theorem 5.24]). *Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on \mathbb{T} and suppose that $f : \mathbb{T} \rightarrow \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the solution of the initial value problem*

$$x^\Delta(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^t e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Theorem 2.43 (See [5, Theorem 5.27]). *Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on \mathbb{T} and suppose that $f : \mathbb{T} \rightarrow \mathbb{R}^n$ is rd-continuous. Then for $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$, the solution of the initial value problem*

$$x^\Delta(t) = -A(t)x^\sigma(t) + f(t), \quad x(t_0) = x_0$$

is given by

$$\begin{aligned} x(t) &= e_{\ominus A}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus A}(t, \tau)f(\tau)\Delta\tau \\ &= e_{A^T}^T(t_0, t) \left[x_0 + \int_{t_0}^t e_{A^T}^T(\tau, t_0)f(\tau)\Delta\tau \right]. \end{aligned}$$

Definition 2.44. A square matrix-valued function A is said to be symmetric if it is equal to its transpose, i.e., $A = A^T$.

Definition 2.45. A symmetric matrix-valued function A is said to be positive definite (denoted $A > 0$) if $x^T A x > 0$ for any nonzero vector x . A symmetric matrix-valued function A is said to be positive semi-definite (denoted $A \geq 0$) if $x^T A x \geq 0$ for any nonzero vector x .

In the next lemma, a Lyapunov function on time scales associated with the autonomous dynamic equation is considered.

$$x^\Delta(t) = Ax(t). \quad (2.23)$$

Definition 2.46. Let $S \in C_{\text{rd}}^1(\mathbb{T}, \mathbb{R}^{n \times n})$ be symmetric. A generalized Lyapunov function is given by

$$x^T(t)S(t)x(t). \quad (2.24)$$

Lemma 2.47. *The derivative of the generalized Lyapunov function is given by*

$$\begin{aligned} (x^T S x)^\Delta(t) &= x^T(t)[A^T S(t) + (I + \mu(t)A^T)S(t)A \\ &\quad + (I + \mu(t)A^T)S^\Delta(t)(I + \mu(t)A)]x(t). \end{aligned} \quad (2.25)$$

Proof. Using the product rule, we have

$$\begin{aligned} (x^T S x)^\Delta(t) &= (x^T S)^\Delta(t)x^\sigma(t) + (x^T S)(t)x^\Delta(t) \\ &= [(x^T)^\Delta(t)S(t) + (x^T)^\sigma(t)S^\Delta(t)]x^\sigma(t) + (x^T S)(t)Ax(t). \end{aligned}$$

Now using the simple useful formula (2.7), we have

$$\begin{aligned}
(x^T S x)^\Delta(t) &= [(x^T(t) A^T S(t) + (x + \mu x^\Delta)^T(t) S^\Delta(t))(x + \mu x^\Delta)(t) + (x^T S)(t) A x(t)] \\
&= [(x^T(t) A^T S(t) + x^T(t) (I + \mu(t) A)^T S^\Delta(t))(I + \mu(t) A) x(t) \\
&\quad + (x^T S)(t) A x(t)] \\
&= x^T(t) [A^T S(t) (I + \mu(t) A) + (I + \mu(t) A)^T S^\Delta(t) (I + \mu(t) A)] x(t) \\
&\quad + (x^T S)(t) A x(t) \\
&= x^T(t) [A^T S(t) + (I + \mu(t) A^T) S(t) A \\
&\quad + (I + \mu(t) A^T) S^\Delta(t) (I + \mu(t) A)] x(t).
\end{aligned}$$

This gives the result as desired. □

3. QUANTUM CALCULUS

The purpose of this section is to outline some of basic definitions and concepts of q -difference equations. Some of the material in this section is contained in monographs by Bangerezako [2], Kac and Cheung [19], and in the books of Bohner and Peterson [5,6].

Definition 3.1. The q -derivative (or Jackson derivative [2]) of a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

We can also use the notation f^Δ for the q -derivative of the function f .

Theorem 3.2. *The q -derivatives of the product and the quotient of f and g are the following, respectively,*

- (i) $D_q(fg) = (D_q f)g + f^\sigma(D_q g) = f(D_q g) + (D_q f)g^\sigma,$
- (ii) $D_q\left(\frac{f}{g}\right) = \frac{(D_q f)g - f(D_q g)}{gg^\sigma} = \frac{(D_q f)g^\sigma - f^\sigma(D_q g)}{gg^\sigma}.$

It follows from Definition 3.1 that the q -derivative of f satisfies

$$f^\sigma(t) = f(qt) = f(t) + (q-1)tD_q f(t) \quad \text{for } t \in \mathbb{T}. \quad (3.1)$$

Example 3.3. The q -derivative of t^n , where n is a positive integer is, $\frac{q^n-1}{q-1}t^{n-1}$, the q -derivative of $\frac{1}{t}$ is $-\frac{1}{qt^2}$, and the q -derivative of $\ln t$ is $\frac{\ln q}{(q-1)t}$.

After having the product rule and quotient rule of q -differentiation, one may wonder about a quantum version of the chain rule. However, there does not exist a general chain rule for q -derivative. An exception is the differentiation of a function of the form $f(u(t))$, where $u = u(t) = \alpha t^\beta$ with α, β are constants. To see how the chain rule applies, we consider

$$D_q[f(u(t))] = D_q[f(\alpha t^\beta)] = \frac{f(\alpha q^\beta t^\beta) - f(\alpha t^\beta)}{(q-1)t}$$

$$\begin{aligned}
&= \frac{f(\alpha q^\beta t^\beta) - f(\alpha t^\beta)}{\alpha q^\beta t^\beta - \alpha t^\beta} \cdot \frac{\alpha q^\beta t^\beta - \alpha t^\beta}{(q-1)t} \\
&= \frac{f(q^\beta u) - f(u)}{(q^\beta - 1)u} \cdot \frac{u(qt) - u(t)}{(q-1)t} \\
&= (D_{q^\beta} f)(u(t)) \cdot D_q u(t).
\end{aligned} \tag{3.2}$$

On the other hand, if for instance $u(t) = t + t^2$ or $u(t) = \sin t$, the quantity $u(qt)$ cannot be expressed in terms of u in a simple manner, and thus it is impossible to have a general chain rule. Next let us consider the q -antiderivative. As in the usual sense, we define the indefinite integral as the following.

Definition 3.4. The indefinite integral of the function f is given by $\int f(t) d_q t = F(t) + C$, where C is an arbitrary constant and F is antiderivative of function f .

Also the definite integral of function f can be defined in the same manner as for \mathbb{Z} .

Definition 3.5. Let $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ and $a, b \in q^{\mathbb{N}_0}$ such that $a < b$. The definite integral of function f is given by

$$\int_a^b f(t) d_q t = (q-1) \sum_{t=a}^{b/q} t f(t).$$

Definition 3.6. The exponential function $e_p(t, t_0)$ on the time scale $\mathbb{T} = q^{\mathbb{N}_0}$, where p is regressive, is given by

$$e_p(t, t_0) = \prod_{s \in [t_0, t)} [1 + (q-1)sp(s)] \text{ if } t > t_0.$$

PAPER

I. FLOQUET THEORY FOR q -DIFFERENCE EQUATIONS

ABSTRACT

In this paper, we introduce ω -periodic functions in quantum calculus and study the first-order linear q -difference vector equation for which its coefficient matrix function is ω -periodic and regressive. Based on the new definition of periodic functions, we establish Floquet theory in quantum calculus.

1. INTRODUCTION

Floquet theory plays an important rôle in many applications such as in linear dynamic systems with periodic coefficient matrix functions. The study of Floquet theory can be found in Kelley and Peterson [5], Hartman [4], and Cronin [3] for \mathbb{R} , and for \mathbb{Z} in Kelley and Peterson [6]. Ahlbrandt and Ridenhour have studied Floquet theory on periodic time scales [1].

In this paper, we are interested to study Floquet theory for q -difference equations, namely dynamic equations on the so-called q -time scale, i.e.,

$$\mathbb{T} := q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0\}, \quad \text{where } q > 1.$$

We present a new definition (see Definition 3.1 below) of periodic functions on the q -time scale and derive some Floquet theory based on the first-order linear equation, called a Floquet q -difference equation,

$$x^\Delta = A(t)x, \tag{1.1}$$

where

$$x^\Delta(t) := \frac{x(qt) - x(t)}{(q-1)t} \quad \text{for } t \in \mathbb{T},$$

A is an ω -periodic matrix function defined as in Definition 3.1 below, and A also is regressive, i.e.,

$$I + (q-1)tA(t) \quad \text{is invertible for all } t \in \mathbb{T},$$

where I is the identity matrix.

2. SOME AUXILIARY RESULTS

The following definitions and theorems are useful to prove the results in Sections 3 and 4 below.

Definition 2.1. Let $m, n \in \mathbb{N}_0$ with $m < n$, and $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$. Then

$$\int_{q^m}^{q^n} f(t) \Delta t := (q - 1) \sum_{k=m}^{n-1} q^k f(q^k).$$

Definition 2.2 (Matrix exponential function). Let $t_0 \in q^{\mathbb{N}_0}$ and A be an $n \times n$ regressive matrix-valued function on $q^{\mathbb{N}_0}$. The unique matrix-valued solution of the initial value problem

$$Y^\Delta = A(t)Y, \quad Y(t_0) = I,$$

where I denotes the $n \times n$ identity matrix, is called the *matrix exponential* function (at t_0), and it is denoted by $e_A(\cdot, t_0)$.

For example, if A is an $n \times n$ regressive matrix-valued function on $q^{\mathbb{N}_0}$ and $s = q^m$, $t = q^n$ with $m, n \in \mathbb{N}_0$ and $m < n$, then

$$\begin{aligned} e_A(t, s) &= \prod_{\tau \in q^{\mathbb{N}_0} \cap [s, t)} [I + (q - 1)\tau A(\tau)] \\ &= \prod_{k=m}^{n-1} [I + (q - 1)q^k A(q^k)], \end{aligned} \tag{2.1}$$

where the matrix product is from the left to the right.

Theorem 2.3 (See [2, Theorem 5.21]). *If A is a matrix-valued function on $q^{\mathbb{N}_0}$, then*

- (i) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$;
- (ii) $e_A(t, s) = e_A^{-1}(s, t)$;
- (iii) $e_A(t, s)e_A(s, r) = e_A(t, r)$.

Theorem 2.4 (Liouville's formula [2, Theorem 5.28]). *Let A be a 2×2 regressive matrix-valued function on $q^{\mathbb{N}_0}$. Assume that X is a matrix-valued solution of*

$$X^\Delta = A(t)X, \quad t \in q^{\mathbb{N}_0}.$$

Then X satisfies

$$\det X(t) = e_{\operatorname{tr} A + (q-1)t \det A}(t, t_0) \det X(t_0), \quad t \in q^{\mathbb{N}_0},$$

where $\operatorname{tr} A$ and $\det A$ denote the trace and the determinant of A , respectively.

In the last section, we shall show an example of a Floquet q -difference equation, whose coefficient matrix function is defined in terms of trigonometric functions on $q^{\mathbb{N}_0}$.

Definition 2.5 (Trigonometric functions). Let p be a function defined on $q^{\mathbb{N}_0}$ and suppose $1 + (q-1)tp(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$. We define the *trigonometric functions* \cos_p and \sin_p by

$$\cos_p := \frac{e_{ip} + e_{-ip}}{2} \quad \text{and} \quad \sin_p := \frac{e_{ip} - e_{-ip}}{2i}.$$

In particular, we have Euler's formula given by

$$e_{ip}(t, t_0) = \cos_p(t, t_0) + i \sin_p(t, t_0),$$

and the identity $[\sin_p(t, t_0)]^2 + [\cos_p(t, t_0)]^2 = 1$ need not hold.

3. PERIODIC FUNCTIONS

Let \mathbb{T} be a periodic time scale with period $T > 0$, i.e., $t + T \in \mathbb{T}$ whenever $t \in \mathbb{T}$. Then a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *periodic* if $f(t + T) = f(t)$ for all $t \in \mathbb{T}$. This definition applies for example to the prominent examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. However, $\mathbb{T} = q^{\mathbb{N}_0}$ is not a periodic time scale. Thus we shall introduce the definition of ω -periodic functions on $q^{\mathbb{N}_0}$ as follows.

Definition 3.1. Let $\omega \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is called ω -periodic if

$$f(t) = q^\omega f(q^\omega t) \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

A first question concerns the geometrical meaning of ω -periodic functions on $q^{\mathbb{N}_0}$. The following theorem and an example below address this issue.

Theorem 3.2. Let f be an ω -periodic function on $q^{\mathbb{N}_0}$ and define

$$c := \int_1^{q^\omega} f(t) \Delta t.$$

Then

$$\int_{q^n}^{q^{n+\omega}} f(t) \Delta t = c \quad \text{for all } n \in \mathbb{N}_0. \quad (3.1)$$

Before we prove Theorem 3.2, let us see an example to better understand the definition of periodic functions on the q -time scale.

Example 3.3. Let $c \in \mathbb{R}$. We define a function $f : 2^{\mathbb{N}_0} \rightarrow \mathbb{R}$ recursively by

$$f(1) := c \quad \text{and} \quad f(2t) := \frac{f(t)}{2} \quad \text{for all } t \in 2^{\mathbb{N}_0}.$$

By Definition 3.1, f is 1-periodic. By Definition 3.1, we have

$$\int_1^2 f(t) \Delta t = f(1) = c$$

and

$$\begin{aligned} \int_{2^n}^{2^{n+1}} f(t) \Delta t &= 2^n f(2^n) = 2^{n-1} \cdot 2 f(2^{n-1} \cdot 2) \\ &= 2^{n-1} f(2^{n-1}) = \dots = 2 f(2) = f(1) = c. \end{aligned}$$

Geometrically, Figure 3.1 shows that the areas under the graph of the function f on the intervals $[2^n, 2^{n+1}]$, $n \in \{0, 1, 2, 3, 4\}$, are all equal to the same constant c .

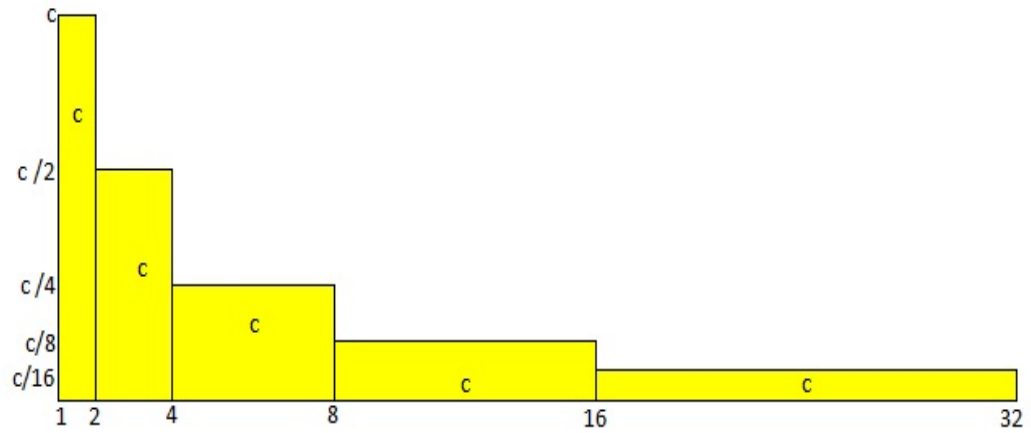


Figure 3.1. The constant area of the rectangles corresponding to the 1-periodic function f on the intervals $[2^n, 2^{n+1}]$, $n \in \{0, 1, 2, 3, 4\}$.

Proof of Theorem 3.2. We prove the statement using the principle of induction. From Definition 2.1 and the assumption, we see that (3.1) holds for $n = 0$. Now let $n \in \mathbb{N}$ and assume that (3.1) holds for $n - 1$, i.e., assume

$$\int_{q^{n-1}}^{q^{n+\omega-1}} f(t) \Delta t = c. \quad (3.2)$$

Using Definition 2.1, Definition 3.1, again Definition 2.1, and (3.2), we obtain

$$\int_{q^n}^{q^{n+\omega}} f(t) \Delta t = (q - 1) \sum_{k=n}^{n+\omega-1} q^k f(q^k)$$

$$\begin{aligned}
&= (q-1) \left\{ \sum_{k=n}^{n+\omega-2} q^k f(q^k) + q^{n+\omega-1} f(q^{n+\omega-1}) \right\} \\
&= (q-1) \left\{ \sum_{k=n}^{n+\omega-2} q^k f(q^k) + q^{n-1} q^\omega f(q^\omega q^{n-1}) \right\} \\
&= (q-1) \left\{ \sum_{k=n}^{n+\omega-2} q^k f(q^k) + q^{n-1} f(q^{n-1}) \right\} \\
&= (q-1) \sum_{k=n-1}^{n+\omega-2} q^k f(q^k) \\
&= \int_{q^{n-1}}^{q^{n+\omega-1}} f(t) \Delta t \\
&= c.
\end{aligned}$$

Hence (3.1) holds for n and the proof is complete. \square

Lemma 3.4. *If B is an ω -periodic and regressive matrix-valued function on $q^{\mathbb{N}_0}$, then*

$$e_B(t, s) = e_B(q^\omega t, q^\omega s) \quad \text{for all } t, s \in q^{\mathbb{N}_0}.$$

Proof. Suppose $s = q^m$ and $t = q^n$ for some $m, n \in \mathbb{N}_0$ with $m < n$. Using (2.1), Definition 3.1, and again (2.1), we obtain

$$\begin{aligned}
e_B(q^\omega t, q^\omega s) &= e_B(q^{\omega+n}, q^{\omega+m}) \\
&= \prod_{k=\omega+m}^{\omega+n-1} \{I + (q-1)q^k B(q^k)\} \\
&= \prod_{k=m}^{n-1} \{I + (q-1)q^{k+\omega} B(q^{k+\omega})\} \\
&= \prod_{k=m}^{n-1} \{I + (q-1)q^k q^\omega B(q^\omega q^k)\} \\
&= \prod_{k=m}^{n-1} \{I + (q-1)q^k B(q^k)\} \\
&= e_B(t, s).
\end{aligned}$$

The proof is complete. \square

Theorem 3.5. *Let $t_0 \in q^{\mathbb{N}_0}$ and $\omega \in \mathbb{N}$. If C is a nonsingular $k \times k$ matrix constant, then there exists an ω -periodic regressive matrix-valued function B on $q^{\mathbb{N}_0}$ such that*

$$e_B(q^\omega t_0, t_0) = C.$$

Proof. Let μ_i be the eigenvalues of C , $1 \leq i \leq k$. For $p \in \{0, 1, 2, \dots, \omega - 2\}$, define

$$R_p := \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_n \end{pmatrix},$$

where either J_i is the 1×1 matrix $J_i = \mu_i$ or

$$J_i = \begin{pmatrix} \mu_i & 1 & 0 & \dots & 0 \\ 0 & \mu_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mu_i & 1 \\ 0 & \dots & 0 & 0 & \mu_i \end{pmatrix},$$

$1 \leq i \leq k$, and define

$$R_{\omega-1} := \frac{1}{(q-1)q^{\omega-1}t_0} \left\{ \prod_{k=0}^{\omega-2} [I + (q-1)q^k t_0 R_k]^{-1} C - I \right\},$$

where I is the identity matrix and $\prod_{k=0}^{\omega-2} [I + (q-1)q^k t_0 R_k]^{-1}$ is the product starting from the right to left. This gives

$$\prod_{k=0}^{\omega-1} [I + (q-1)q^k t_0 R_k] = C,$$

where $\prod_{k=0}^{\omega-1} [I + (q-1)q^k t_0 R_k]$ is the product starting from the left to right. Moreover, R_p are regressive for all $p \in \{0, 1, 2, \dots, \omega-1\}$. We define

$$B(q^{\omega m+j} t_0) := \frac{R_j}{q^{\omega m}} \quad \text{for all } j \in \{0, 1, 2, \dots, \omega-1\} \quad \text{and all } m \in \mathbb{N}_0.$$

Therefore B is ω -periodic and regressive on $q^{\mathbb{N}_0}$ and

$$e_B(q^{\omega} t_0, t_0) = \prod_{k=0}^{\omega-1} [I + (q-1)q^k t_0 B(q^k t_0)] = C,$$

where $\prod_{k=0}^{\omega-1} [I + (q-1)q^k t_0 B(q^k t_0)]$ is the product starting from the left to right. \square

4. FLOQUET THEORY

In this section, we consider the Floquet q -difference equation (1.1) where A is a regressive and ω -periodic matrix-valued function.

Lemma 4.1. *Let $t_0 \in q^{\mathbb{N}_0}$ and suppose x is a solution of the Floquet q -difference equation (1.1) satisfying the boundary condition*

$$x(t_0) = q^\omega x(q^\omega t_0).$$

Then x is ω -periodic.

Proof. Define a function f on $q^{\mathbb{N}_0}$ by

$$f(t) := q^\omega x(q^\omega t) - x(t) \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

Then $f(t_0) = 0$ and

$$\begin{aligned} f^\Delta(t) &= \frac{f(qt) - f(t)}{(q-1)t} \\ &= \frac{q^\omega x(qq^\omega t) - x(qt) - q^\omega x(q^\omega t) + x(t)}{(q-1)t} \\ &= q^\omega q^\omega \frac{x(qq^\omega t) - x(q^\omega t)}{(q-1)q^\omega t} - \frac{x(qt) - x(t)}{(q-1)t} \\ &= q^\omega q^\omega x^\Delta(q^\omega t) - x^\Delta(t) \\ &= q^\omega q^\omega A(q^\omega t)x(q^\omega t) - A(t)x(t) \\ &= A(t)[q^\omega x(q^\omega t) - x(t)] \\ &= A(t)f(t). \end{aligned}$$

By unique solvability of the initial value problem $f^\Delta = A(t)f$, $f(t_0) = 0$, we conclude $f(t) = 0$ for all $t \in q^{\mathbb{N}_0}$. By Definition 3.1, x is ω -periodic. \square

As usual, we call a matrix-valued function Φ a *fundamental matrix* of the Floquet q -difference equation (1.1) provided it solves (1.1) such that $\Phi(t)$ is nonsingular for all

$t \in q^{\mathbb{N}_0}$. The following results gives a representation for any fundamental matrix of the Floquet q -difference equation (1.1).

Theorem 4.2. *Suppose Φ is a fundamental matrix for the Floquet q -difference equation (1.1). Define the matrix-valued function Ψ by*

$$\Psi(t) := q^\omega \Phi(q^\omega t), \quad t \in q^{\mathbb{N}_0}.$$

Then Ψ is also a fundamental matrix for (1.1). Furthermore, there exist an ω -periodic and regressive matrix-valued function B and an ω -periodic matrix-valued function P such that

$$\Phi(t) = P(t)e_B(t, t_0) \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

Proof. Assume Φ is a fundamental matrix for (1.1) and define Ψ as in the statement of the theorem. Then

$$\begin{aligned} \Psi^\Delta(t) &= \frac{\Psi(qt) - \Psi(t)}{(q-1)t} \\ &= \frac{q^\omega \Phi(q^\omega qt) - q^\omega \Phi(q^\omega t)}{(q-1)t} \\ &= q^\omega q^\omega \frac{\Phi(qq^\omega t) - \Phi(q^\omega t)}{(q-1)q^\omega t} \\ &= q^\omega q^\omega \Phi^\Delta(q^\omega t) \\ &= q^\omega q^\omega A(q^\omega t) \Phi(q^\omega t) \\ &= q^\omega A(t) \Phi(q^\omega t) \\ &= A(t) \Psi(t). \end{aligned}$$

Since $\det \Psi(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$, Ψ is a fundamental matrix for (1.1). Furthermore, define now the nonsingular constant matrix C by

$$C := \Phi^{-1}(t_0) \Psi(t_0).$$

The function D defined by $D(t) = \Psi(t) - \Phi(t)C$, $t \in q^{\mathbb{N}_0}$, satisfies $D(t_0) = 0$ and

$$D^\Delta(t) = \Psi^\Delta(t) - \Phi^\Delta(t)C = A(t)\Psi(t) - A(t)\Phi(t)C = A(t)D(t)$$

and thus, by unique solvability of this initial value problem, we conclude

$$q^\omega \Phi(q^\omega t) = \Psi(t) = \Phi(t)C \quad \text{for all } t \in q^{\mathbb{N}_0}. \quad (4.1)$$

By Theorem 3.5, there exists an ω -periodic and regressive matrix-valued function B such that

$$e_B(q^\omega t_0, t_0) = C. \quad (4.2)$$

Now define the matrix-valued function P by

$$P(t) := \Phi(t)e_B^{-1}(t, t_0), \quad t \in q^{\mathbb{N}_0}.$$

Obviously, P is a nonsingular matrix-valued function on $q^{\mathbb{N}_0}$. Using (4.1), Theorem 2.3 (i), (ii), (4.2), and Lemma 3.4, we obtain

$$\begin{aligned} q^\omega P(q^\omega t) &= q^\omega \Phi(q^\omega t)e_B^{-1}(q^\omega t, t_0) \\ &= \Phi(t)Ce_B(t_0, q^\omega t) \\ &= \Phi(t)Ce_B(t_0, q^\omega t_0)e_B(q^\omega t_0, q^\omega t) \\ &= \Phi(t)e_B(t_0, t) \\ &= \Phi(t)e_B^{-1}(t, t_0) \\ &= P(t) \end{aligned}$$

for all $t \in q^{\mathbb{N}_0}$, i.e., P is ω -periodic. □

Theorem 4.3. *Suppose Φ , P , and B are as in Theorem 4.2. Then x solves the Floquet q -difference equation (1.1) if and only if y given by $y(t) = P^{-1}(t)x(t)$, $t \in q^{\mathbb{N}_0}$, solves $y^\Delta = B(t)y$.*

Proof. Assume x solves (1.1). Then, as can be seen again by unique solvability of initial value problems as in the proof of Theorem 4.2, we have

$$x(t) = \Phi(t)x_0 \quad \text{for all } t \in q^{\mathbb{N}_0}, \quad \text{where } x_0 := \Phi^{-1}(t_0)x(t_0).$$

Define y by $y(t) = P^{-1}(t)x(t)$, $t \in q^{\mathbb{N}_0}$. Then

$$y(t) = P^{-1}(t)\Phi(t)x_0 = P^{-1}(t)P(t)e_B(t, t_0)x_0 = e_B(t, t_0)x_0,$$

which solves $y^\Delta = B(t)y$. Conversely, assume y solves $y^\Delta = B(t)y$ and define x by $x(t) = P(t)y(t)$, $t \in q^{\mathbb{N}_0}$. Again by unique solvability of initial value problems, we have

$$y(t) = e_B(t, t_0)y_0 \quad \text{for all } t \in q^{\mathbb{N}_0}, \quad \text{where } y_0 := e_B(t_0, t)P(t_0)y(t_0).$$

It follows that

$$x(t) = P(t)y(t) = P(t)e_B(t, t_0)y_0 = \Phi(t)y_0,$$

which solves (1.1). □

Definition 4.4. Let Φ be a fundamental matrix for the Floquet q -difference equation (1.1). The eigenvalues of $q^\omega \Phi^{-1}(1)\Phi(q^\omega)$ are called the *Floquet multipliers* of the Floquet q -difference equation (1.1).

Remark 4.5. Since fundamental matrices for the Floquet q -difference equation (1.1) are not unique, we shall show that the Floquet multipliers are well defined. Let Φ and Ψ be any fundamental matrices for (1.1) and let

$$C := q^\omega \Phi^{-1}(1)\Phi(q^\omega) \quad \text{and} \quad D := q^\omega \Psi^{-1}(1)\Psi(q^\omega).$$

We show that C and D have the same eigenvalues. Since Φ and Ψ are fundamental matrices of (1.1), we see as in the proof of Theorem 4.2 that there exists a nonsingular constant matrix M such that

$$\Psi(t) = \Phi(t)M \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

It follows that

$$D = q^\omega \Psi^{-1}(1) \Psi(q^\omega) = q^\omega M^{-1} \Phi^{-1}(1) \Phi(q^\omega) M = M^{-1} C M.$$

Therefore C and D are similar matrices, and thus they have the same eigenvalues. Hence, Floquet multipliers are well defined.

Remark 4.6. Note also that the proof of Theorem 4.2 shows that the matrix-valued function

$$q^\omega \Phi^{-1}(t) \Phi(q^\omega t) = \Phi^{-1}(t) \Psi(t) \equiv \Phi^{-1}(1) \Psi(1) = q^\omega \Phi^{-1}(1) \Phi(q^\omega)$$

does not depend on $t \in q^{\mathbb{N}_0}$, and therefore Floquet multipliers of the Floquet q -difference equation (1.1) are also equal to the eigenvalues of $q^\omega \Phi^{-1}(t) \Phi(q^\omega t)$, where $t \in q^{\mathbb{N}_0}$ is arbitrary.

Theorem 4.7. *The number μ_0 is a Floquet multiplier of the Floquet q -difference equation (1.1) if and only if there exists a nontrivial solution x of (1.1) such that $q^\omega x(q^\omega t) = \mu_0 x(t)$ for all $t \in q^{\mathbb{N}_0}$.*

Proof. Assume μ_0 is a Floquet multiplier of (1.1). Let $t \in q^{\mathbb{N}_0}$. By Remark 4.6, μ_0 is an eigenvalue of $C := q^\omega \Phi^{-1}(t) \Phi(q^\omega t)$, where Φ is a fundamental matrix of (1.1). Let x_0 be an eigenvector corresponding to the eigenvalue μ_0 , i.e., we have $Cx_0 = \mu_0 x_0$. Define x by $x(t) = \Phi(t)x_0$ for all $t \in q^{\mathbb{N}_0}$. Then x is a nontrivial solution of (1.1) and

$$q^\omega x(q^\omega t) = q^\omega \Phi(q^\omega t)x_0 = \Phi(t)Cx_0 = \Phi(t)\mu_0 x_0 = \mu_0 x(t).$$

Conversely, assume that there exists a nontrivial solution x of (1.1) such that $q^\omega x(q^\omega t) = \mu_0 x(t)$ for all $t \in q^{\mathbb{N}_0}$. Let Ψ be a fundamental matrix of (1.1). Then $x(t) = \Psi(t)y_0$ for

all $t \in q^{\mathbb{N}_0}$ and some nonzero constant vector y_0 . Furthermore, $q^\omega \Psi(q^\omega t)$ is a fundamental matrix of (1.1). Hence

$$q^\omega x(q^\omega t) = \mu_0 x(t) \quad \text{and} \quad q^\omega \Psi(q^\omega t) y_0 = \mu_0 \Psi(t) y_0.$$

Since $q^\omega \Psi(q^\omega t) = \Psi(t)D$, where $D := q^\omega \Psi^{-1}(1)\Psi(q^\omega)$ and $\Psi(t)Dy_0 = \Psi(t)\mu_0 y_0$, it follows that $Dy_0 = \mu_0 y_0$, and hence μ_0 is an eigenvalue of D . \square

Remark 4.8. By Theorem 4.7, the Floquet q -difference equation (1.1) has an ω -periodic solution if and only if $\mu_0 = 1$ is a Floquet multiplier.

5. APPLICATION AND AN EXAMPLE

Let p be defined by

$$p(q^{2n}t) := \frac{1}{q^{2n}} \quad \text{and} \quad p(q^{2n+1}t) := \frac{2}{q^{2n+1}} \quad \text{for all } t \in q^{\mathbb{N}_0} \quad \text{and all } n \in \mathbb{N}_0.$$

Then p is a 2-periodic regressive function on $q^{\mathbb{N}_0}$. Define

$$A(t) := \begin{pmatrix} 0 & \frac{1}{t} \cos_p(q^2t, t) \\ \frac{1}{t} \sin_p(q^2t, t) & 0 \end{pmatrix}, \quad \text{for all } t \in q^{\mathbb{N}_0} \quad (5.1)$$

with the given 2-periodic regressive function p . We apply Lemma 3.4 to show that the coefficient matrix-valued function A is 2-periodic:

$$\begin{aligned} q^2 A(q^2t) &= q^2 \begin{pmatrix} 0 & \frac{1}{q^2t} \cos_p(q^4t, q^2t) \\ \frac{1}{q^2t} \sin_p(q^4t, q^2t) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{e_{ip}(q^4t, q^2t) + e_{-ip}(q^4t, q^2t)}{2t} \\ \frac{e_{ip}(q^4t, q^2t) - e_{-ip}(q^4t, q^2t)}{2ti} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{t} \cos_p(q^2t, t) \\ \frac{1}{t} \sin_p(q^2t, t) & 0 \end{pmatrix} \\ &= A(t). \end{aligned}$$

The solution of the Floquet q -difference equation $x^\Delta = A(t)x$, where A is defined as in (5.1), satisfying the initial condition $x(1) = x_0$, is $x(t) = e_A(t, 1)x_0$, $t \in q^{\mathbb{N}_0}$. If μ_1 and μ_2 are eigenvalues corresponding to the constant matrix

$$C := q^2 e_A^{-1}(1, 1) e_A(q^2, 1) = q^2 e_A(q^2, 1),$$

then by applying Liouville's formula (Theorem 2.4), we get

$$\mu_1 \mu_2 = \det C = \det q^2 e_A(q^2, 1) = q^4 \det e_A(q^2, 1)$$

$$\begin{aligned}
&= q^4 e_{\operatorname{tr} A + (q-1)t \det A}(q^2, 1) \det e_A(1, 1) \\
&= q^4 e_{\frac{(1-q) \sin_p \cos_p}{t}}(q^2, 1).
\end{aligned}$$

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II. THE BEVERTON–HOLT q -DIFFERENCE EQUATION

ABSTRACT

The Beverton–Holt model is a classical population model which has been considered in the literature for the discrete-time case. Its continuous-time analogue is the well-known logistic model. In this paper, we consider a quantum calculus analogue of the Beverton–Holt equation. We use a recently introduced concept of periodic functions in quantum calculus in order to study the existence of periodic solutions of the Beverton–Holt q -difference equation. Moreover, we present proofs of quantum calculus versions of two so-called Cushing–Henson conjectures.

1. INTRODUCTION

The Beverton–Holt difference equation has wide applications in population growth and is given by

$$x_{n+1} = \frac{\nu K_n x_n}{K_n + (\nu - 1)x_n}, \quad n \in \mathbb{N}_0, \quad (1.1)$$

where $\nu > 1$, $K_n > 0$, and $x_0 > 0$. We call the sequence K the *carrying capacity* and ν the *inherent growth rate* (see Cushing and Henson [7]). The periodically forced Beverton–Holt equation, which is obtained by letting the carrying capacity be a periodic positive sequence K_n with period $\omega \in \mathbb{N}$, i.e., $K_{n+\omega} = K_n$ for all $n \in \mathbb{N}_0$, has been treated with the methods found in [1, 5, 8, 9]. For the Beverton–Holt dynamic equation on time scales, one article has been presented by Bohner and Warth [6]. In [6], a general Beverton–Holt equation is given, which reduces to (1.1) in the discrete case and to the well-known logistic equation in the continuous case.

In this paper, we are studying a quantum calculus version of the Beverton–Holt equation, namely, a Beverton–Holt q -difference equation. Using a recently by the authors introduced concept of periodic functions in quantum calculus (see [3]), we are interested to seek periodic solutions of the Beverton–Holt q -difference equation given by

$$x^\Delta(t) = a(t)x^\sigma(t) \left(1 - \frac{x(t)}{K(t)} \right), \quad (1.2)$$

where

$$a(t) = \frac{\alpha}{t}, \quad (1.3)$$

$K(t) = q^\omega K(q^\omega t)$ for all $t \in \mathbb{T} = q^{\mathbb{N}_0}$, $\alpha \in \mathbb{R}$, $\omega \in \mathbb{N}$,

$$x^\Delta(t) = \frac{x(qt) - x(t)}{(q-1)t}, \quad x^\sigma(t) = x(qt), \quad t \in \mathbb{T}.$$

By the definition of periodic functions on the q -time scale, i.e., on $q^{\mathbb{N}_0}$ (see Definition 2.3 below), a is 1-periodic and K is ω -periodic. We approach the periodic solutions of the Beverton–Holt q -difference equation (1.2) by some strategies presented in Section 3. In Sections 4 and 5, we formulate and prove the first and the second Cushing–Henson conjectures on the q -time scale, respectively.

2. SOME AUXILIARY RESULTS

Definition 2.1. We say that a function $p : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is *regressive* provided

$$1 + (q - 1)tp(t) \neq 0 \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

The set of all regressive functions will be denoted by \mathcal{R} .

Definition 2.2 (Exponential function). Let $p \in \mathcal{R}$ and $t_0 \in q^{\mathbb{N}_0}$. The *exponential function* $e_p(\cdot, t_0)$ on $q^{\mathbb{N}_0}$ is defined by

$$e_p(t, t_0) = \prod_{s \in [t_0, t)} [1 + (q - 1)sp(s)] \quad \text{for } t > t_0.$$

Definition 2.3 (See [3]). A function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is called ω -*periodic* if

$$f(t) = q^\omega f(q^\omega t) \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

Theorem 2.4 (See [4, Theorem 2.36]). *If $p \in \mathcal{R}$, then*

- (i) $e_0(t, s) = 1$ and $e_p(t, t) = 1$;
- (ii) $e_p(t, s) = \frac{1}{e_p(s, t)}$;
- (iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (iv) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (v) $\left(\frac{1}{e_p(\cdot, s)}\right)^\Delta(t) = -\frac{p(t)}{e_p(\sigma(t), s)}.$

The *integral* on $q^{\mathbb{N}_0}$ is defined as follows.

Definition 2.5. Let $m, n \in \mathbb{N}_0$ with $m < n$, and $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$. Then

$$\int_{q^m}^{q^n} f(t) \Delta t := (q - 1) \sum_{k=m}^{n-1} q^k f(q^k).$$

Theorem 2.6 (Integration by parts, see [4, Theorem 1.77]). *For $a, b \in q^{\mathbb{N}_0}$ and $f, g : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$, we have*

$$\int_a^b f^\sigma(t) g^\Delta(t) \Delta t = f(b)g(b) - f(a)g(a) + \int_a^b f^\Delta(t) g(t) \Delta t$$

and

$$\int_a^b f(t) g^\Delta(t) \Delta t = f(b)g(b) - f(a)g(a) + \int_a^b f^\Delta(t) g^\sigma(t) \Delta t.$$

Theorem 2.7 (Jensen's inequality, see [10, Theorem 2.2]). *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose $g, h : ([a, b] \cap q^{\mathbb{N}_0}) \rightarrow (c, d)$ and $\int_a^b |h(s)| \Delta s > 0$. If $F \in C((c, d), \mathbb{R})$ is convex, then*

$$F \left(\frac{\int_a^b |h(s)| g(s) \Delta s}{\int_a^b |h(s)| \Delta s} \right) \leq \frac{\int_a^b |h(s)| F(g(s)) \Delta s}{\int_a^b |h(s)| \Delta s}.$$

If F is strictly convex, then " \leq " can be replaced by " $<$ ".

3. THE BEVERTON–HOLT EQUATION

Throughout we assume

$$\alpha \neq \frac{1}{q-1} \quad \text{and} \quad \alpha \neq -1,$$

i.e.,

$$\lambda := 1 - (q-1)\alpha \quad \text{satisfies} \quad \lambda \neq 0 \quad \text{and} \quad \lambda \neq q.$$

This implies that

$$-a \in \mathcal{R} \quad \text{and} \quad e_{-a}(t, s) = \lambda^{\log_q(\frac{t}{s})} \quad \text{for all} \quad t, s \in q^{\mathbb{N}_0}.$$

In the dynamic equation (1.2), we substitute

$$x := \frac{1}{u}.$$

Then, using the quotient rule [4, Theorem 1.20 (v)], (1.2) becomes

$$u^\Delta(t) = -a(t)u(t) + \frac{a(t)}{K(t)}. \quad (3.1)$$

The general solution of (3.1) is given [4, Theorem 2.77] by

$$u(t) = e_{-a}(t, t_0)u(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s, \quad t \in q^{\mathbb{N}_0}, \quad (3.2)$$

where $t_0 \in q^{\mathbb{N}_0}$. Now, we require an ω -periodic solution \bar{x} of (1.2). This means that \bar{x} satisfies $\bar{x}(t) = q^\omega \bar{x}(q^\omega t)$ for all $t \in q^{\mathbb{N}_0}$. This implies that a solution $\bar{u} = \frac{1}{\bar{x}}$ of (3.1) satisfies

$$q^\omega \bar{u}(t) = \bar{u}(q^\omega t) \quad \text{for all} \quad t \in q^{\mathbb{N}_0}. \quad (3.3)$$

Lemma 3.1. *If (3.1) has a solution \bar{u} satisfying (3.3), then*

$$\bar{u}(t_0) = \frac{1}{q^\omega \lambda^{-\omega} - 1} \int_{t_0}^{q^\omega t_0} e_{-a}(t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s.$$

Proof. Assume (3.1) has a solution \bar{u} satisfying (3.3). Then

$$\begin{aligned} \bar{u}(t_0) &= q^{-\omega} \bar{u}(q^\omega t_0) \\ &= q^{-\omega} e_{-a}(q^\omega t_0, t_0) \bar{u}(t_0) + q^{-\omega} \int_{t_0}^{q^\omega t_0} e_{-a}(q^\omega t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\ &= \frac{q^{-\omega}}{1 - q^{-\omega} e_{-a}(q^\omega t_0, t_0)} \int_{t_0}^{q^\omega t_0} e_{-a}(q^\omega t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\ &= \frac{q^{-\omega}}{1 - q^{-\omega} \lambda^\omega} e_{-a}(q^\omega t_0, t_0) \int_{t_0}^{q^\omega t_0} e_{-a}(t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\ &= \frac{\lambda^\omega}{q^\omega - \lambda^\omega} \int_{t_0}^{q^\omega t_0} e_{-a}(t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\ &= \frac{1}{q^\omega \lambda^{-\omega} - 1} \int_{t_0}^{q^\omega t_0} e_{-a}(t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \end{aligned}$$

Thus \bar{u} satisfies the required initial condition. □

4. THE FIRST CUSHING–HENSON CONJECTURE

Now we state and prove the first Cushing–Henson conjecture for the Beverton–Holt q -difference equation (1.2).

Conjecture 4.1 (First Cushing–Henson conjecture). *The Beverton–Holt q -difference model (1.2) with an ω -periodic carrying capacity K has a unique ω -periodic solution \bar{x} that globally attracts all solutions.*

Using (3.2) and Lemma 3.1, the solution \bar{u} of (3.1) can be written as

$$\begin{aligned}\bar{u}(t) &= e_{-a}(t, t_0)\bar{u}(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\ &= \frac{1}{q^\omega \lambda^{-\omega} - 1} \int_{t_0}^{q^\omega t_0} e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\ &= \int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{sK(s)} \Delta s,\end{aligned}\tag{4.1}$$

where

$$h(t, s) := e_{-a}(t, \sigma(s))(\beta + \chi(t, s))\alpha\tag{4.2}$$

with

$$\beta := \frac{1}{q^\omega \lambda^{-\omega} - 1} \quad \text{and} \quad \chi(t, s) := \begin{cases} 1 & \text{if } s < t \\ 0 & \text{if } s \geq t. \end{cases}$$

Theorem 4.2. *Define $\bar{x} := \frac{1}{\bar{u}}$, where \bar{u} is given in (4.1). Then \bar{x} is an ω -periodic solution of the Beverton–Holt q -difference equation (1.2).*

Proof. To verify that the solution \bar{u} of (3.1) indeed satisfies (3.3), we only prove that $f^\Delta(t) = -a(t)f(t)$ and $f(t_0) = 0$, where f is defined by $f(t) = q^{-\omega}\bar{u}(q^\omega t) - \bar{u}(t)$ for all $t \in q^{\mathbb{N}_0}$. Hence $\bar{u}(q^\omega t) = q^\omega \bar{u}(t)$ for all $t \in q^{\mathbb{N}_0}$ which implies that the solution \bar{x} of (1.2) is ω -periodic. \square

Theorem 4.3. *The solution \bar{x} of (1.2) given in Theorem 4.2 is globally attractive.*

Proof. Let x be any solution of the equation (1.2). We have

$$\begin{aligned}
|x(t) - \bar{x}(t)| &= \left| \frac{1}{e_{-a}(t, t_0)u(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s} \right. \\
&\quad \left. - \frac{1}{e_{-a}(t, t_0)\bar{u}(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s} \right| \\
&= \left| \frac{1}{\frac{e_{-a}(t, t_0)}{x(t_0)} + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s} - \frac{1}{\frac{e_{-a}(t, t_0)}{\bar{x}(t_0)} + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s} \right| \\
&= \frac{\left| \frac{1}{\bar{x}(t_0)} - \frac{1}{x(t_0)} \right| |e_{-a}(t, t_0)|}{\left| \frac{e_{-a}(t, t_0)}{x(t_0)} + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \right| \left| \frac{e_{-a}(t, t_0)}{\bar{x}(t_0)} + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \right|} \\
&\leq \frac{\left| \frac{1}{\bar{x}(t_0)} - \frac{1}{x(t_0)} \right| |e_{-a}(t, t_0)|}{\left(\int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \right)^2} \\
&\leq \frac{\|K\|_\infty^2 \left| \frac{1}{\bar{x}(t_0)} - \frac{1}{x(t_0)} \right| |e_{-a}(t, t_0)|}{(1 - e_{-a}(t, t_0))^2},
\end{aligned}$$

which due to [2, Theorem 2] tends to zero as $t \rightarrow \infty$. □

5. THE SECOND CUSHING–HENSON CONJECTURE

Now we state and prove the second Cushing–Henson conjecture for the Beverton–Holt q -difference equation (1.2).

Conjecture 5.1 (Second Cushing–Henson conjecture). *The average of the ω -periodic solution \bar{x} of (1.2) is strictly less than the average of the ω -periodic carrying capacity K times the constant $1 + \frac{1}{\alpha}$.*

In order to prove the second Cushing–Henson conjecture, we use the following series of auxiliary results.

Lemma 5.2. *We have*

$$\int_u^v e_{-a}(t, \sigma(s)) \Delta s = \frac{ve_{-a}(t, v) - ue_{-a}(t, u)}{1 + \alpha}, \quad (5.1)$$

where a is given by (1.3).

Proof. Using Theorem 2.4 (ii), (iv) and Theorem 2.6, we get

$$\begin{aligned} \int_u^v e_{-a}(t, \sigma(s)) \Delta s &= \int_u^v \frac{e_{-a}^{\Delta_s}(t, s)}{a(s)} \Delta s \\ &= \frac{1}{\alpha q} \int_u^v \sigma(s) e_{-a}^{\Delta_s}(t, s) \Delta s \\ &= \frac{1}{\alpha q} \left\{ ve_{-a}(t, v) - ue_{-a}(t, u) - \int_u^v \frac{1}{e_{-a}(s, t)} \Delta s \right\} \\ &= \frac{1}{\alpha q} \left\{ ve_{-a}(t, v) - ue_{-a}(t, u) - \int_u^v \lambda e_{-a}(t, \sigma(s)) \Delta s \right\} \\ &= \frac{ve_{-a}(t, v) - ue_{-a}(t, u)}{\alpha q + \lambda} \\ &= \frac{ve_{-a}(t, v) - ue_{-a}(t, u)}{1 + \alpha}, \end{aligned}$$

which shows (5.1). □

Lemma 5.3. *We have*

$$\int_u^v \frac{e_{-a}(t, s)}{t^2} \Delta t = \frac{q}{1 + \alpha} \left\{ \frac{e_{-a}(u, s)}{u} - \frac{e_{-a}(v, s)}{v} \right\} \quad (5.2)$$

where a is given by (1.3).

Proof. Using Theorem 2.4 and Theorem 2.6, we get

$$\begin{aligned}
\int_u^v \frac{e_{-a}(t, s)}{t^2} \Delta t &= \frac{-1}{\alpha} \int_u^v \frac{-a(t)e_{-a}(t, s)}{t} \Delta t \\
&= \frac{-1}{\alpha} \int_u^v \frac{e_{-a}^{\Delta t}(t, s)}{t} \Delta t \\
&= \frac{-1}{\alpha} \left\{ \frac{e_{-a}(v, s)}{v} - \frac{e_{-a}(u, s)}{u} - \int_u^v e_{-a}(\sigma(t), s) \left(\frac{-1}{t\sigma(t)} \right) \Delta t \right\} \\
&= \frac{-1}{\alpha} \left\{ \frac{e_{-a}(v, s)}{v} - \frac{e_{-a}(u, s)}{u} \right\} - \frac{\lambda}{q\alpha} \int_u^v \frac{e_{-a}(t, s)}{t^2} \\
&= \frac{q}{\alpha + 1} \left\{ \frac{e_{-a}(u, s)}{u} - \frac{e_{-a}(v, s)}{v} \right\},
\end{aligned}$$

which shows (5.2). □

Lemma 5.4. *We have*

$$\int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{t^2} \Delta t = \frac{\alpha}{(\alpha + 1)s}, \tag{5.3}$$

where h is given by (4.2).

Proof. Using Lemma 5.3 and $\beta q^\omega \lambda^{-\omega} - \beta - 1 = 0$, we obtain

$$\begin{aligned}
\int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{t^2} \Delta t &= \alpha \beta \int_{t_0}^{q^\omega t_0} \frac{e_{-a}(t, \sigma(s))}{t^2} \Delta t + \alpha \int_{\sigma(s)}^{q^\omega t_0} \frac{e_{-a}(t, \sigma(s))}{t^2} \Delta t \\
&\stackrel{(5.2)}{=} \frac{\alpha q}{\alpha + 1} \left\{ \beta \left(\frac{e_{-a}(t_0, \sigma(s))}{t_0} - \frac{e_{-a}(q^\omega t_0, \sigma(s))}{q^\omega t_0} \right) + \frac{1}{qs} - \frac{e_{-a}(q^\omega t_0, \sigma(s))}{q^\omega t_0} \right\} \\
&= \frac{\alpha q}{\alpha + 1} \left\{ \frac{1}{qs} + \frac{e_{-a}(q^\omega t_0, \sigma(s))}{q^\omega t_0} (\beta q^\omega \lambda^{-\omega} - \beta - 1) \right\} \\
&= \frac{\alpha}{(\alpha + 1)s},
\end{aligned}$$

which shows (5.3). □

Lemma 5.5. *We have*

$$\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s = \frac{\alpha t}{1 + \alpha}, \tag{5.4}$$

where h is given by (4.2).

Proof. Using Lemma 5.2 and $\beta q^\omega \lambda^{-\omega} - \beta - 1 = 0$, we obtain

$$\begin{aligned}
\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s &= \alpha \beta \int_{t_0}^{q^\omega t_0} e_{-a}(t, \sigma(s)) \Delta s + \alpha \int_{t_0}^t e_{-a}(t, \sigma(s)) \Delta s \\
&\stackrel{(5.1)}{=} \alpha \beta \left(\frac{q^\omega t_0 e_{-a}(t, q^\omega t_0) - t_0 e_{-a}(t, t_0)}{1 + \alpha} \right) + \alpha \left(\frac{t - t_0 e_{-a}(t, t_0)}{1 + \alpha} \right) \\
&= \frac{\alpha}{1 + \alpha} \{ t + t_0 e_{-a}(t, t_0) (\beta q^\omega \lambda^{-\omega} - \beta - 1) \} \\
&= \frac{\alpha t}{1 + \alpha},
\end{aligned}$$

which shows (5.4). □

Theorem 5.6. *Let \bar{x} be the unique ω -periodic solution of (1.2). If $\omega \neq 1$, then*

$$\frac{1}{\omega} \int_{t_0}^{q^\omega t_0} \bar{x}(t) \Delta t < \left(1 + \frac{1}{\alpha} \right) \left\{ \frac{1}{\omega} \int_{t_0}^{q^\omega t_0} K(t) \Delta t \right\}. \quad (5.5)$$

Proof. Since K is ω -periodic with $\omega \neq 1$, $tK(t)$ cannot be a constant. In addition, $F(x) = \frac{1}{x}$ is strictly convex. Thus we may use Jensen's inequality (Theorem 2.7) for the single inequality in the forthcoming calculation to obtain

$$\begin{aligned}
\int_{t_0}^{q^\omega t_0} \bar{x}(t) \Delta t &= \int_{t_0}^{q^\omega t_0} \frac{1}{\bar{u}(t)} \Delta t \\
&= \int_{t_0}^{q^\omega t_0} \frac{1}{\frac{\int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{sK(s)} \Delta s}{\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s}} \Delta t \\
&= \int_{t_0}^{q^\omega t_0} F \left(\frac{\int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{sK(s)} \Delta s}{\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s} \right) \frac{1}{\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s} \Delta t \\
&< \int_{t_0}^{q^\omega t_0} \frac{\int_{t_0}^{q^\omega t_0} h(t, s) F \left(\frac{1}{sK(s)} \right) \Delta s}{\left(\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s \right)^2} \Delta t \\
&= \int_{t_0}^{q^\omega t_0} \frac{\int_{t_0}^{q^\omega t_0} h(t, s) sK(s) \Delta s}{\left(\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s \right)^2} \Delta t \\
&\stackrel{(5.4)}{=} \int_{t_0}^{q^\omega t_0} \frac{\int_{t_0}^{q^\omega t_0} h(t, s) sK(s) \Delta s}{\left(\frac{\alpha t}{1 + \alpha} \right)^2} \Delta t \\
&= \left(\frac{1 + \alpha}{\alpha} \right)^2 \int_{t_0}^{q^\omega t_0} \int_{t_0}^{q^\omega t_0} \frac{h(t, s) sK(s)}{t^2} \Delta s \Delta t
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1+\alpha}{\alpha} \right)^2 \int_{t_0}^{q^{\omega}t_0} sK(s) \int_{t_0}^{q^{\omega}t_0} \frac{h(t,s)}{t^2} \Delta t \Delta s \\
&\stackrel{(5.3)}{=} \left(\frac{1+\alpha}{\alpha} \right)^2 \int_{t_0}^{q^{\omega}t_0} sK(s) \frac{\alpha}{(\alpha+1)s} \Delta s \\
&= \frac{1+\alpha}{\alpha} \int_{t_0}^{q^{\omega}t_0} K(s) \Delta s,
\end{aligned}$$

which shows (5.5). The proof is done. \square

Theorem 5.7. *If K is 1-periodic, then we have equality in (5.5), i.e.,*

$$\frac{1}{\omega} \int_{t_0}^{q^{\omega}t_0} \bar{x}(t) \Delta t = \left(1 + \frac{1}{\alpha} \right) \left\{ \frac{1}{\omega} \int_{t_0}^{q^{\omega}t_0} K(t) \Delta t \right\}. \quad (5.6)$$

Proof. Since K is 1-periodic, we have

$$K(t) = \frac{C}{t} \quad \text{for some } C > 0.$$

Now it is easy to check that \bar{x} given by

$$\bar{x}(t) := \frac{1+\alpha}{\alpha} K(t) = \frac{(1+\alpha)C}{\alpha t}$$

is 1-periodic and satisfies

$$\bar{x}^{\Delta}(t) = a(t) \bar{x}^{\sigma}(t) \left(1 - \frac{\bar{x}(t)}{K(t)} \right) \quad \text{for all } t \in q^{\mathbb{N}_0}.$$

Hence, \bar{x} is the unique 1-periodic solution of (1.2). Thus (5.6) holds. \square

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III. STABILITY FOR HAMILTONIAN q -DIFFERENCE SYSTEMS

ABSTRACT

In this paper, we study stability of q -difference Hamiltonian systems with or without parameter λ in quantum calculus. Based on a new definition of periodic functions in quantum calculus, we obtain q -analogues of classical stability results in the continuous case.

1. INTRODUCTION

Stability analysis of the linear Hamiltonian system

$$x'(t) = \mathcal{J}\mathcal{H}(t)x(t), \quad (1.1)$$

where $\mathcal{H}(t) = \mathcal{H}^*(t) = \mathcal{H}(T + t)$ and $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, has been studied by Kreĭn and Jakubovič [5], and for the discrete version of (1.1) it has been found in Răsvan [6] and [3]. In this paper, we are interested in the study of q -difference Hamiltonian systems on the q -time scale $\mathbb{T} := q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$,

$$x^\Delta(t) = \mathcal{J}\mathcal{H}(t) [\mathcal{M}^T \mathcal{M}x^\sigma(t) + \mathcal{M}\mathcal{M}^T x(t)], \quad (1.2)$$

where x^Δ is as given in Definition 2.3,

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},$$

I denotes the identity matrix, $\mathcal{H}(t)$ is a Hermitian matrix-valued function, and

$$\mathcal{H}(t) = q^\omega \mathcal{H}(q^\omega t) \quad \text{for every } t \in \mathbb{T}.$$

An equivalent equation of (1.2) is

$$x^\Delta(t) = S(t)x(t), \quad (1.3)$$

where

$$S(t) = (I - \mu(t)\mathcal{J}\mathcal{H}(t)\mathcal{M}^T\mathcal{M})^{-1} \mathcal{J}\mathcal{H}(t), \quad (1.4)$$

$$S^*(t)\mathcal{J} + \mathcal{J}S(t) + \mu(t)S^*(t)\mathcal{J}S(t) = 0, \quad (1.5)$$

and $\mu(t) = (q - 1)t$ is the graininess function for all $t \in \mathbb{T}$, see [2]. Also, if the Hermitian matrix $\mathcal{H}(t)$ is given by

$$\mathcal{H}(t) = \begin{pmatrix} A(t) & B^*(t) \\ B(t) & C(t) \end{pmatrix}$$

for all $t \in \mathbb{T}$, then (1.2) becomes

$$x(qt) = \begin{pmatrix} D(t)B(t) + I & D(t)C(t) \\ -\mu(t)A(t)\{I + D(t)B(t)\} & -\mu(t)\{A(t)D(t)C(t) + B^*(t)\} + I \end{pmatrix} x(t), \quad (1.6)$$

where $D(t) = \mu(t)(I - \mu(t)B(t))^{-1}$ for all $t \in \mathbb{T}$. We see that the solution of (1.2) can be constructed from (1.6) if the matrix $I - \mu(t)B(t)$ is invertible for all $t \in \mathbb{T}$. Since the stability of (1.2) is connected with the eigenvalues of its fundamental matrix at the end point q^ω of the period, we shall discuss this issue in Sections 2 and 3. For convenience, we give the following definition.

Definition 1.1. We call (1.2) *Hamiltonian* if it has the complex coefficients and $U(q^\omega)$ is \mathcal{J} -unitary, i.e., $U^*(q^\omega)\mathcal{J}U(q^\omega) = \mathcal{J}$. Moreover, (1.2) is called *canonical* if it has real coefficients and $U(q^\omega)$ is \mathcal{J} -orthogonal (or symplectic), i.e., $U^T(q^\omega)\mathcal{J}U(q^\omega) = \mathcal{J}$, where $U(q^\omega)$ represents a fundamental matrix at the end point q^ω of the period.

2. PRELIMINARIES AND AUXILIARY RESULTS

Definition 2.1 (Matrix exponential function, Bohner and Peterson [2]). Let $t_0 \in \mathbb{T}$ and A be a regressive matrix-valued function on \mathbb{T} , i.e., $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$. The unique matrix-valued solution of the initial value problem

$$Y^\Delta = A(t)Y, \quad Y(t_0) = I,$$

where I is identity matrix, is called the *matrix exponential* function (at t_0), and it is denoted by $e_A(\cdot, t_0)$.

Remark 2.2. Since the matrix-valued function S given in (1.3) is regressive, i.e.,

$$I + (q - 1)tS(t) \neq 0 \quad \text{for all } t \in \mathbb{T},$$

we obtain

$$e_S(t, 1) = \prod_{\tau \in \mathbb{T} \cap [1, t)} [I + (q - 1)\tau S(\tau)] \quad \text{for all } t \in \mathbb{T}. \quad (2.1)$$

Before we prove that the fundamental matrix $e_S(t, t_0)$ of (1.3) is \mathcal{J} -unitary for all $t \in \mathbb{T}$, Definition 2.3 and Theorem 2.4 are given as follows.

Definition 2.3 (See [4]). Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function. The expression

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q - 1)t}, \quad t \in q^{\mathbb{N}_0},$$

is called the *q-derivative*

The *q*-derivatives of the product and quotient of $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are given by

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma$$

and

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma},$$

where $f^\sigma = f \circ \sigma$, $g^\sigma = g \circ \sigma$, $\sigma(t) = qt$ for all $t \in \mathbb{T}$.

Theorem 2.4 (Bohner and Peterson [2]). *If A is a matrix-valued function on \mathbb{T} , then*

- (i) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$;
- (ii) $e_A(t, s) = e_A^{-1}(s, t)$;
- (iii) $e_A(t, s)e_A(s, r) = e_A(t, r)$;
- (iv) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$.

Lemma 2.5. *If S is as in (1.4), then*

$$e_S^*(t, t_0)\mathcal{J}e_S(t, t_0) = \mathcal{J}.$$

Proof. Let $f(t) = e_S^*(t, t_0)\mathcal{J}e_S(t, t_0)$. Obviously $f(t_0) = \mathcal{J}$. By applying the product rule of the q -derivative and (i), (iii), and (iv) from Theorem 2.4, we have

$$\begin{aligned} f^\Delta(t) &= (e_S^\Delta(t, t_0))^* \mathcal{J}e_S(\sigma(t), t_0) + e_S^*(t, t_0)\mathcal{J}e_S^\Delta(t, t_0) \\ &= (S(t)e_S(t, t_0))^* \mathcal{J}(I + \mu(t)S(t))e_S(t, t_0) + e_S^*(t, t_0)\mathcal{J}S(t)e_S(t, t_0) \\ &= e_S^*(t, t_0) [S^*(t)\mathcal{J} + \mathcal{J}S(t) + \mu(t)S^*(t)\mathcal{J}S(t)] e_S(t, t_0) = 0, \end{aligned}$$

because $S^*(t)\mathcal{J} + \mathcal{J}S(t) + \mu(t)S^*(t)\mathcal{J}S(t) = 0$. This implies $f(t) = \mathcal{J}$. □

Let $U(t) := e_S(t, 1)$ for all $t \in \mathbb{T}$. If we let $\{\rho_i\}$ be the spectrum of $U(q^\omega)$, i.e., the set of all eigenvalues of $U(q^\omega)$, then the spectra of $U^*(q^\omega)$ and $U^{-1}(q^\omega)$ are the sets $\{\bar{\rho}_i\}$ and $\left\{\frac{1}{\rho_i}\right\}$, respectively. Since $U^*(q^\omega) = \mathcal{J}U^{-1}(q^\omega)\mathcal{J}^{-1}$, the sets $\{\bar{\rho}_i\}$ and $\left\{\frac{1}{\rho_i}\right\}$ coincide. Consequently, so do the sets $\{\rho_i\}$ and $\left\{\frac{1}{\rho_i}\right\}$. Thus, if ρ is an eigenvalue of $U(q^\omega)$, then so is $\frac{1}{\rho}$. The complex numbers ρ and $\frac{1}{\rho}$ are symmetric with respect to the unit circle. In general, we have the following.

Lemma 2.6. *For any matrix A such that $A^* \mathcal{J} A = \mathcal{J}$, the spectrum of A is symmetric about the unit circle.*

In case of the canonical system, the spectrum is symmetric with respect to both the real axis and the unit circle, i.e., nonreal eigenvalues that are not on the unit circle are partitioned into $\rho, \bar{\rho}, \frac{1}{\rho}$, and $\frac{1}{\bar{\rho}}$.

Definition 2.7. (i) The fundamental matrix $U(q^\omega)$ at the end point of the period is called the *monodromy* matrix.

(ii) The eigenvalues of the monodromy matrix $U(q^\omega)$, i.e., the roots ρ of the characteristic equation $\det(U(q^\omega) - \rho I) = 0$ are called the *multipliers* of (1.2).

Theorem 2.8 (Lyapunov–Poincaré). *The multipliers of the Hamiltonian (1.2) (or the canonical (1.2)) equation allowing for their multipliers and the structure of the elementary divisors, are symmetric about the unit circle.*

We must consider the elementary divisors of the monodromy matrix $U(q^\omega)$ instead of their spectrums to obtain more precise information. Let $\{(\rho - \rho_i)^{m_i}\}$ be the set of the elementary divisors of the monodromy matrix $U(q^\omega)$, i.e., $\det(U(q^\omega) - \rho I) = 0$. Then the set of the elementary divisors of the monodromy matrix $U^*(q^\omega)$ is $\{(\rho - \bar{\rho}_i)^{m_i}\}$. Repeating the above arguments, we see that the symmetry properties of the spectrum remain valid when the elementary divisors are taken into consideration. In other words, if ρ_0 is an eigenvalue of the matrix $U(q^\omega)$ that is not on the unit circle (not on the imaginary axis) and the corresponding elementary divisors are

$$(\rho - \rho_0)^{m_1}, \quad (\rho - \rho_0)^{m_2}, \quad \dots, \quad (\rho - \rho_0)^{m_i},$$

then the number $\rho_1 = \frac{1}{\bar{\rho}_0}$ is also an eigenvalue with the elementary divisors

$$(\rho - \rho_1)^{m_1}, \quad (\rho - \rho_1)^{m_2}, \quad \dots, \quad (\rho - \rho_1)^{m_i},$$

Thus the Lyapunov–Poincaré theorem holds.

At the beginning, we mentioned periodic functions on the time scale \mathbb{T} , and the following is the formal definition.

Definition 2.9 (Bohner and Chieochan [1]). A function $f : \mathbb{T} \rightarrow \mathbb{R}$ with

$$f(t) = q^\omega f(q^\omega t) \quad \text{for all } t \in \mathbb{T}$$

is called ω -periodic.

Definition 2.10. Let $s, t \in \mathbb{T}$ with $t > s$ and $f : \mathbb{T} \rightarrow \mathbb{R}$. Then the q -integral is defined by

$$\int_s^t f(\tau) \Delta\tau := (q - 1) \sum_{\tau \in [s, t) \cap \mathbb{T}} \tau f(\tau).$$

Remark 2.11. For equation (1.2) such that $\mathcal{H}(t)$ is a complex symmetric matrix for all $t \in \mathbb{T}$ and \mathcal{H} is ω -periodic, we have the following results:

- (i) $U^T(t) \mathcal{J} U(t) = \mathcal{J}$ for all $t \in \mathbb{T}$.
- (ii) If a number ρ is a multiplier of (1.2) with the complex symmetric \mathcal{H} , then so is $\frac{1}{\rho}$ and they have the same structure of the elementary divisors. Then its spectrum is skew-symmetric with respect to the unit circle.

Lemma 2.12. *The matrix S given as (1.4) is ω -periodic.*

Proof. Since $\mu(q^\omega t) = q^\omega \mu(t)$ for all $t \in \mathbb{T}$ and \mathcal{H} is ω -periodic, we have

$$\begin{aligned} q^\omega S(q^\omega t) &= q^\omega (I - \mu(q^\omega t) \mathcal{J} \mathcal{H}(q^\omega t) \mathcal{M}^T \mathcal{M})^{-1} \mathcal{J} \mathcal{H}(q^\omega t) \\ &= (I - q^\omega \mu(t) \mathcal{J} \mathcal{H}(q^\omega t) \mathcal{M}^T \mathcal{M})^{-1} \mathcal{J} \mathcal{H}(t) \\ &= (I - \mu(t) \mathcal{J} \mathcal{H}(t) \mathcal{M}^T \mathcal{M})^{-1} = S(t) \end{aligned}$$

for all $t \in \mathbb{T}$, i.e., the matrix-valued function S is ω -periodic. □

Lemma 2.13. *If x is a solution, U is a fundamental matrix, and the number ρ is a multiplier of (1.2), then for any $t \in \mathbb{T}$,*

- (i) $U(t)$ is ω -periodic if and only if $U(q^\omega) = q^{-\omega} I$,

(ii) $x(q^\omega t) = \rho x(t)$ for any $t \in \mathbb{T}$, where U is ω -periodic.

Proof. First, we prove (i). Let

$$X^\Delta(t) = S(t)X(t), \quad X(1) = I$$

be the Hamiltonian matrix equation (with initial value) which is equivalent to (1.3). Then its fundamental matrix $U(t)$ is $e_S(t, 1)$. By the definition of the matrix exponential on the q -time scale and Lemma 2.12, (i) holds. Next, we prove (ii). Let (ρ, x_0) be an eigenpair of the matrix $U(q^\omega)$. Since $x(t) = U(t)x_0$ for all $t \in \mathbb{T}$ and by applying (i),

$$x(q^\omega t) = U(q^\omega t)x_0 = q^{-\omega}U(t)x_0 = q^{-\omega}U(t) \cdot q^\omega U(q^\omega)x_0 = U(t)\rho x_0 = \rho x(t).$$

The proof is complete. □

The following theorem is used in the proof of Theorem 4.6 in Section 4.

Theorem 2.14 (Smith–McMillan). *Let $U(s) = [U_{ik}(s)]$ be an $m \times m$ matrix-valued function, where $U_{ik}(s)$ are the rational scalar functions $U(s) = \frac{P(s)}{l(s)}$, where $P(s)$ is an $m \times m$ polynomial matrix of rank r , and $l(s)$ is the least common multiple of the denominator of all elements $U_{ik}(s)$. Then $U(s)$ is equivalent to the matrix $U^{SM}(s)$ given by*

$$U^{SM}(s) = \text{diag} \left(\frac{\varepsilon_1(s)}{\delta_1(s)}, \frac{\varepsilon_2(s)}{\delta_2(s)}, \dots, \frac{\varepsilon_r(s)}{\delta_r(s)}, 0, \dots, 0 \right),$$

where $(\varepsilon_i(s), \delta_i(s))$ is a pair of monic and coprime polynomials for $i = 1, 2, \dots, r$. Furthermore, $\varepsilon_i(s)$ is a factor of $\varepsilon_{i+1}(s)$ and $\delta_i(s)$ is a factor of $\delta_{i-1}(s)$.

The matrix U^{SM} is called the *Smith–McMillan form* of the matrix U .

3. STRONG AND WEAK STABILITY

The terms of weak and strong stability of Hamiltonian q -difference equations are defined as follows.

Definition 3.1. (i) The equation (1.2) is called *weakly stable* or *stable* if all its solutions are bounded on \mathbb{T} .

(ii) The equation (1.2) is called *strongly stable* if it is weakly stable and there is $\delta > 0$ such that all solutions of any equation

$$x^\Delta(t) = \mathcal{J}\mathcal{H}_1(t) [\mathcal{M}^T \mathcal{M}x^\sigma(t) + \mathcal{M}\mathcal{M}^T x(t)],$$

where \mathcal{H}_1 is ω -periodic and Hermitian,

$$\int_1^{q^\omega} |\mathcal{H}(t) - \mathcal{H}_1(t)| \Delta t < \delta,$$

and the notation $|\cdot|$ means a matrix norm, are bounded on \mathbb{T} .

We use the following terminologies given in Kreĭn [5], Halanay and Răsvan [3], or Răsvan [6].

- (a) A vector $v \in \mathbb{C}^n$ is *plus*, *minus*, or *null* vector,
- (b) A matrix U is \mathcal{J} -*decreasing* or \mathcal{J} -*increasing*,
- (c) An eigenvalue ρ is of *first*, *second*, or *mixed (indefinite) kind*.

Proposition 3.2. *All solutions of (1.2) are bounded on \mathbb{T} if and only if all multipliers of (1.2) have modulus one and they are simple type.*

Proof. First, we assume that all solutions of (1.2) are bounded on \mathbb{T} , and ρ_1 and ρ_2 are any multipliers of (1.2) which are not on the unit circle. Since $U(q^\omega)$ is \mathcal{J} -unitary and $\rho_1 \overline{\rho_2} \neq 1$, $\eta^* \mathcal{J} \xi = 0$ for any $\xi \in L_{\rho_1}, \eta \in L_{\rho_2}$, where L_{ρ_i} are the eigensubspaces corresponding to the multipliers ρ_i ($i = 1, 2$). This implies $U^*(q^\omega) \mathcal{J} U(q^\omega) = 0$ which

gives a contradiction as $U(q^\omega)$ is \mathcal{J} -unitary, and breaks up the bounded property of all solutions. Thus all multipliers have modulus one.

Suppose there is a multiplier ρ of the equation (1.2) which is not simple. Then there are two linearly independent solutions u and v of (1.2) corresponding to ρ such that

$$U(q^\omega)u = \rho u \quad \text{and} \quad U(q^\omega)v = \rho v + u.$$

Thus

$$u^* \mathcal{J} v = u^* U^*(q^\omega) \mathcal{J} U(q^\omega) v = \bar{\rho} u^* \mathcal{J} (\rho v + u) = |\rho|^2 u^* \mathcal{J} v + \bar{\rho} u^* \mathcal{J} u,$$

and then $u^* \mathcal{J} u = 0$ because $|\rho| = 1$. This implies the Hamiltonian equation (1.2) is not stable on \mathbb{T} , which gives a contradiction. Hence ρ is simple.

Conversely, we assume that all multipliers of (1.2) have modulus one and they are of simple type. Thus, for each multiplier ρ and its corresponding eigenvector u , $iu^* \mathcal{J} u$ preserves the same sign on its eigensubspace while vanishing only at $u = 0$. By applying Theorem 2.8 and matrix theory [5], this implies all solutions of (1.2) are bounded on \mathbb{T} . \square

Generally speaking, if the equation (1.2) is stable, then the matrix $U(q^\omega)$ is said to be of *stable type*, i.e., all its eigenvalues have modulus one and they are simple.

Theorem 3.3 (Kreĭn [5, Theorem 1.2]). *If a \mathcal{J} -unitary matrix U is of stable type, then so are all \mathcal{J} -unitary matrices V in some δ -neighborhood $|U - V| < \delta$ of it.*

Theorem 3.4 (Yakubovich and Starzhinskii [7]). *Let U_0 be a \mathcal{J} -unitary matrix having a definite eigenvalue ρ_0 on the unit circle, i.e., the eigenvalue ρ is the first or second kind. Then*

- (i) *the eigenvalue ρ_0 is simple,*
- (ii) *for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any \mathcal{J} -unitary matrix U satisfying the inequality $|U - U_0| < \delta$, all eigenvalues in the neighborhood $|\rho - \rho_0| < \varepsilon$ lie on the unit circle and they are simple.*

Proposition 3.5 (Kreĭn [5]). *A sufficient condition for strong stability of the equation (1.2) is that all its multipliers lie on the unit circle and they are definite.*

Theorem 3.6 (See Bohner and Peterson [2]). *Let $y \in C_{rd}$, $p \in \mathcal{R}^+$, $p \geq 0$, and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leq \alpha + \int_{t_0}^t y(\tau)p(\tau)\Delta\tau \quad \text{for all } t > t_0$$

implies

$$y(t) \leq \alpha e_p(t, t_0) \quad \text{for all } t > t_0.$$

Lemma 3.7. *If f and g are any nonnegative functions on \mathbb{T} , then*

$$\int_1^{q^\omega} f(t)g(t)\Delta t \leq \frac{1}{q-1} \left(\int_1^{q^\omega} f(t)\Delta t \right) \left(\int_1^{q^\omega} g(t)\Delta t \right).$$

Proof. We have

$$\begin{aligned} \left(\int_1^{q^\omega} f(t)\Delta t \right) \left(\int_1^{q^\omega} g(t)\Delta t \right) &= \left(\sum_{i=0}^{\omega-1} \mu(q^i) f(q^i) \right) \left(\sum_{j=0}^{\omega-1} \mu(q^j) g(q^j) \right) \\ &= \sum_{i=0}^{\omega-1} \sum_{j=0}^{\omega-1} \mu(q^i) f(q^i) \mu(q^j) g(q^j) \\ &\geq \sum_{k=0}^{\omega-1} \mu^2(q^k) f(q^k) g(q^k) \\ &\geq (q-1) \sum_{k=0}^{\omega-1} \mu(q^k) f(q^k) g(q^k) \\ &= (q-1) \int_1^{q^\omega} f(t)g(t)\Delta t. \end{aligned}$$

Dividing by $q-1$ completes the proof. □

Theorem 3.8. *If the Hamiltonian equation (1.2) is stable, then there exists some $\delta > 0$ such that all Hamiltonian equations with the corresponding ω -periodic Hermitian $\tilde{\mathcal{H}}$ matrices which satisfy*

$$\int_1^{q^\omega} |\mathcal{H}(t) - \tilde{\mathcal{H}}(t)| \Delta t < \delta,$$

are also stable.

Remark 3.9. By Definition 3.1 and Theorem 3.8, the equation (1.2) is strongly stable.

Proof of Theorem 3.8. Let us consider the Hamiltonian equations

$$x^\Delta(t) = S(t)x(t), \quad w^\Delta(t) = \tilde{S}(t)w(t), \quad (3.1)$$

where

$$S(t) = (I - \mu(t)\mathcal{J}\mathcal{H}(t)\mathcal{M}^T\mathcal{M})^{-1}\mathcal{J}\mathcal{H}(t)$$

and

$$\tilde{S}(t) = (I - \mu(t)\mathcal{J}\tilde{\mathcal{H}}(t)\mathcal{M}^T\mathcal{M})^{-1}\mathcal{J}\tilde{\mathcal{H}}(t)$$

for all $t \in \mathbb{T}$. Next we shall estimate the upper bound of $|S(t) - \tilde{S}(t)|$ for all $t \in \mathbb{T} \cap [1, q^\omega]$, where $|\cdot|$ means a matrix norm. For convenience, we shall write S instead of $S(t)$ and do the same for the other functions. We have

$$\begin{aligned} \int_1^{q^\omega} |S - \tilde{S}| \Delta t &= \int_1^{q^\omega} |(I - \mu\mathcal{J}\mathcal{H}\mathcal{M}^T\mathcal{M})^{-1}\mathcal{J}\mathcal{H} - (I - \mu\mathcal{J}\tilde{\mathcal{H}}\mathcal{M}^T\mathcal{M})^{-1}\mathcal{J}\tilde{\mathcal{H}}| \Delta t \\ &= \int_1^{q^\omega} |(I - \mu\mathcal{J}\mathcal{H}\mathcal{M}^T\mathcal{M})^{-1}\mathcal{J}(\mathcal{H} - \tilde{\mathcal{H}}) + (I - \mu\mathcal{J}\mathcal{H}\mathcal{M}^T\mathcal{M})^{-1}\mathcal{J}\tilde{\mathcal{H}} \\ &\quad - (I - \mu\mathcal{J}\tilde{\mathcal{H}}\mathcal{M}^T\mathcal{M})^{-1}\mathcal{J}\tilde{\mathcal{H}}| \Delta t \\ &\leq |\mathcal{J}| \int_1^{q^\omega} \{ |(I - \mu\mathcal{J}\mathcal{H}\mathcal{M}^T\mathcal{M})^{-1}| |\mathcal{H} - \tilde{\mathcal{H}}| + |\tilde{\mathcal{H}}| |(I - \mu\mathcal{J}\mathcal{H}\mathcal{M}^T\mathcal{M})^{-1} \\ &\quad - (I - \mu\mathcal{J}\tilde{\mathcal{H}}\mathcal{M}^T\mathcal{M})^{-1}| \} \Delta t \\ &\leq |\mathcal{J}| \left(\int_1^{q^\omega} |(I - \mu\mathcal{J}\mathcal{H}\mathcal{M}^T\mathcal{M})^{-1}| \Delta t \right) \left(\int_1^{q^\omega} |\mathcal{H} - \tilde{\mathcal{H}}| \Delta t \right) \\ &\quad + |\mathcal{J}| \left(\int_1^{q^\omega} |\tilde{\mathcal{H}}| \Delta t \right) \left(\int_1^{q^\omega} |(I - \mu\mathcal{J}\mathcal{H}\mathcal{M}^T\mathcal{M})^{-1} - (I - \mu\mathcal{J}\tilde{\mathcal{H}}\mathcal{M}^T\mathcal{M})^{-1}| \Delta t \right) \\ &\leq m_1 \int_1^{q^\omega} |\mathcal{H} - \tilde{\mathcal{H}}| \Delta t + m_2, \end{aligned}$$

where

$$m_2 = |\mathcal{J}| \left(\int_1^{q^\omega} |\tilde{\mathcal{H}}| \Delta t \right) \left(\int_1^{q^\omega} |(I - \mu\mathcal{J}\mathcal{H}\mathcal{M}^T\mathcal{M})^{-1} - (I - \mu\mathcal{J}\tilde{\mathcal{H}}\mathcal{M}^T\mathcal{M})^{-1}| \Delta t \right)$$

and

$$m_1 = |\mathcal{J}| \int_1^{q^\omega} |(I - \mu \mathcal{J} \mathcal{H} \mathcal{M}^T \mathcal{M})^{-1}| \Delta t.$$

Assume U and V are any fundamental solution matrices for (3.1), respectively. Let us consider a matrix Hamiltonian equation

$$Y^\Delta(t) = S(t)Y + F(t), \quad Y(1) = I, \quad (3.2)$$

where S is given as (3.1) and I is the identity matrix. The matrix solution of equation (3.2) is given by (see Bohner and Peterson [2])

$$\begin{aligned} Y(t) &= e_S(t, 1) + \int_1^t e_S(t, \sigma(\tau)) F(\tau) \Delta \tau \\ &= e_S(t, 1) + e_S(t, 1) \int_1^t e_S^{-1}(\sigma(\tau), 1) F(\tau) \Delta \tau. \end{aligned}$$

If $F(t) = (\tilde{S}(t) - S(t))V(t)$ for all $t \in \mathbb{T}$, then we have

$$\begin{aligned} V(t) &= e_S(t, 1) + e_S(t, 1) \int_1^t e_S^{-1}(\sigma(\tau), 1) (\tilde{S}(\tau) - S(\tau)) V(\tau) \Delta \tau \\ &= U(t) + U(t) \int_1^t U^{-1}(\sigma(\tau)) (\tilde{S}(\tau) - S(\tau)) V(\tau) \Delta \tau. \end{aligned}$$

Thus

$$|V(q^\omega) - U(q^\omega)| \leq |U(q^\omega)| \int_1^{q^\omega} |U^{-1}(\sigma(\tau))| |\tilde{S}(\tau) - S(\tau)| |V(\tau)| \Delta \tau.$$

Since $U^\Delta(t) = S(t)U(t)$ and $U(1) = I$, we have

$$U(t) = I + \int_1^t S(\tau)U(\tau) \Delta \tau \quad \text{and} \quad |U(t)| \leq \alpha_1 + \int_1^t |S(\tau)| |U(\tau)| \Delta \tau,$$

where $\alpha_1 = |I|$. By Theorem 3.6,

$$|U(t)| \leq \alpha_1 e_{|S|}(t, 1) \quad \text{for all } t \in \mathbb{T}.$$

This gives

$$|U(q^\omega)| \leq \alpha_1 e_{|S|}(q^\omega, 1) \leq \alpha_1 e^{\int_1^{q^\omega} |S(\tau)| \Delta \tau}.$$

Also in the same way, for all $t \in \mathbb{T}$,

$$\begin{aligned} |V(t)| &\leq \alpha_2 e^{\int_1^t |\tilde{S}(\tau)| \Delta\tau} \leq \alpha_2 e^{\int_1^t |\tilde{S}(\tau) - S(\tau)| \Delta\tau + \int_1^t |S(\tau)| \Delta\tau}, \\ |U^{-1}(t)| &\leq \alpha_3 e^{\int_1^t |S(\tau)| \Delta\tau}, \end{aligned}$$

where α_2 and α_3 are some real numbers. Hence by the inequality (3.2) and the previous results,

$$\begin{aligned} |V(q^\omega) - U(q^\omega)| &\leq \alpha_1 \alpha_2 \alpha_3 e^{3 \int_1^{q^\omega} |S(\tau)| \Delta\tau + \int_1^{q^\omega} |\tilde{S}(\tau) - S(\tau)| \Delta\tau} \int_1^{q^\omega} |\tilde{S}(\tau) - S(\tau)| \Delta\tau \\ &< \alpha_1 \alpha_2 \alpha_3 (m_1 \delta + m_2) e^{3 \int_1^{q^\omega} |S(\tau)| \Delta\tau + m_1 \delta + m_2} \end{aligned}$$

provided

$$\int_1^{q^\omega} |\mathcal{H}(\tau) - \tilde{\mathcal{H}}(\tau)| \Delta\tau < \delta$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Now $U(q^\omega)$ is \mathcal{J} -unitary of the stable type and if

$$|V(q^\omega) - U(q^\omega)| < \varepsilon$$

for given $\varepsilon > 0$, then there is $\delta = \delta_\varepsilon$ by choosing from the inequality

$$\alpha_1 \alpha_2 \alpha_3 (m_1 \delta + m_2) e^{3\beta + m_1 \delta + m_2} < \varepsilon$$

such that

$$\int_1^{q^\omega} |\mathcal{H}(\tau) - \tilde{\mathcal{H}}(\tau)| \Delta\tau < \delta,$$

where $\beta = \int_1^{q^\omega} |S(\tau)| \Delta\tau$. □

4. BVP FOR HAMILTONIAN EQUATIONS WITH A PARAMETER

The following equation is the boundary value problem for Hamiltonian equation (1.2) with parameter λ :

$$x^\Delta(t) = \lambda \mathcal{J} \mathcal{H}(t) [\mathcal{M}^T \mathcal{M} x^\sigma(t) + \mathcal{M} \mathcal{M}^T x(t)], \quad x(1) = q^\omega x(q^\omega). \quad (4.1)$$

Since sometimes we discuss the equation (4.1) without the given boundary condition, let us denote that equation by (4.1)*. We call the number λ satisfying (4.1) (or (4.1)*) an *eigenvalue* or *characteristic value*.

Definition 4.1. We say that \mathcal{H} belongs to the class $\mathcal{P}(q^\omega)$, and write $\mathcal{H} \in \mathcal{P}(q^\omega)$, if

- (i) $\mathcal{H}(t) \geq 0$ for all $t \in [1, q^\omega] \cap \mathbb{T}$, and
- (ii) $\int_1^{q^\omega} \mathcal{H}(t) \Delta t > 0$.

The conditions (i) and (ii) mean that for any vector $\eta \in \mathbb{C}^n$, $\eta \neq 0$, $\eta^* \mathcal{H}(t) \eta \geq 0$ for all $t \in [1, q^\omega] \cap \mathbb{T}$, and

$$\int_1^{q^\omega} \eta^* \mathcal{H}(t) \eta \Delta t > 0.$$

Lemma 4.2. Suppose $x(t; \lambda) = \begin{pmatrix} y(t; \lambda) \\ z(t; \lambda) \end{pmatrix}$ is a solution of equation (4.1)* and shortly

write $x = \begin{pmatrix} y \\ z \end{pmatrix}$ for convenience. If $\mathcal{H} \in \mathcal{P}(q^\omega)$, then for any solution $x = x(t; \lambda) \neq 0$ of (4.1)*, the equality

$$\begin{pmatrix} y \\ z \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix} \Big|_{t=q^\omega} - \begin{pmatrix} y \\ z \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix} \Big|_{t=1} = (\bar{\lambda} - \lambda) \int_1^{q^\omega} \begin{pmatrix} y^\sigma \\ z \end{pmatrix}^* \mathcal{H} \begin{pmatrix} y^\sigma \\ z \end{pmatrix} \Delta t \quad (4.2)$$

holds, where

$$\int_1^{q^\omega} \begin{pmatrix} y^\sigma \\ z \end{pmatrix}^* \mathcal{H} \begin{pmatrix} y^\sigma \\ z \end{pmatrix} \Delta t > 0. \quad (4.3)$$

Proof. Since,

$$\begin{aligned}
\left[\begin{pmatrix} y \\ z \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix} \right]^{\Delta_t} &= \begin{pmatrix} y \\ z \end{pmatrix}^{*\Delta_t} \mathcal{J} \begin{pmatrix} y^\sigma \\ z^\sigma \end{pmatrix} + \begin{pmatrix} y \\ z \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix}^{\Delta_t} \\
&= \begin{pmatrix} y \\ z \end{pmatrix}^{*\Delta_t} \mathcal{J} \begin{pmatrix} y^\sigma \\ z + \mu z^{\Delta_t} \end{pmatrix} + \begin{pmatrix} y^\sigma - \mu y^{\Delta_t} \\ z \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix}^{\Delta_t} \\
&= \begin{pmatrix} y \\ z \end{pmatrix}^{*\Delta_t} \mathcal{J} \begin{pmatrix} y^\sigma \\ z \end{pmatrix} + \begin{pmatrix} y \\ z \end{pmatrix}^{*\Delta_t} \mathcal{J} \begin{pmatrix} 0 \\ \mu z^{\Delta_t} \end{pmatrix} + \begin{pmatrix} y^\sigma \\ z \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix}^{\Delta_t} - \begin{pmatrix} \mu y^{\Delta_t} \\ 0 \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix}^{\Delta_t} \\
&= \begin{pmatrix} y \\ z \end{pmatrix}^{*\Delta_t} \mathcal{J} \begin{pmatrix} y^\sigma \\ z \end{pmatrix} + \begin{pmatrix} y^\sigma \\ z \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y \\ z \end{pmatrix}^{\Delta_t} \\
&= \left(\lambda \mathcal{J} \mathcal{H} \begin{pmatrix} y^\sigma \\ z \end{pmatrix} \right)^* \mathcal{J} \begin{pmatrix} y^\sigma \\ z \end{pmatrix} + \begin{pmatrix} y^\sigma \\ z \end{pmatrix}^* \mathcal{J} \cdot \lambda \mathcal{J} \mathcal{H} \begin{pmatrix} y^\sigma \\ z \end{pmatrix} \\
&= (\bar{\lambda} - \lambda) \begin{pmatrix} y^\sigma \\ z \end{pmatrix}^* \mathcal{H} \begin{pmatrix} y^\sigma \\ z \end{pmatrix},
\end{aligned}$$

by integrating both sides of the above equality, we obtain (4.2). Since $\mathcal{H} \in \mathcal{P}(q^\omega)$, this gives the inequality (4.3). \square

Remark 4.3. (i) A number λ is a root of the equation $\det(U(q^\omega; \lambda) - q^{-\omega}I) = 0$, where $U(q^\omega; \lambda)$ is the fundamental matrix solution at the end point q^ω of the period for (4.1).

(ii) $\lambda = 0$ is not eigenvalue of (4.1) because

$$\begin{aligned}
U(t; \lambda) &= e_S(t; \lambda) = \prod_{\tau \in [1, t)} [I + \mu(\tau)S(\tau; \lambda)] \\
&= \prod_{\tau \in [1, t)} [I + \lambda^2 \mu(\tau)(I - \lambda \mu(\tau) \mathcal{J} \mathcal{H}(\tau) \mathcal{M}^T \mathcal{M})^{-1} \mathcal{J} \mathcal{H}(\tau)]
\end{aligned}$$

and

$$\det(U(q^\omega; \lambda = 0) - q^{-\omega}I) = \det(I - q^{-\omega}I) = (1 - q^{-\omega})^n \neq 0,$$

where the number n is the dimension of the matrix considered.

Theorem 4.4. *If $\mathcal{H} \in \mathcal{P}(q^\omega)$, then all eigenvalues of (4.1) are real.*

Proof. From the left side of (4.2) and with the boundary condition $x(1) = q^\omega x(q^\omega)$, we obtain

$$\begin{aligned}
& \begin{pmatrix} y(q^\omega; \lambda) \\ z(q^\omega; \lambda) \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y(q^\omega; \lambda) \\ z(q^\omega; \lambda) \end{pmatrix} - \begin{pmatrix} y(1; \lambda) \\ z(1; \lambda) \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y(1; \lambda) \\ z(1; \lambda) \end{pmatrix} \\
&= (1 - q^\omega) \begin{pmatrix} y(q^\omega; \lambda) \\ z(q^\omega; \lambda) \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y(q^\omega; \lambda) \\ z(q^\omega; \lambda) \end{pmatrix} \\
&= (1 - q^\omega) \begin{pmatrix} y^*(q^\omega; \lambda) & z^*(q^\omega; \lambda) \end{pmatrix} \begin{pmatrix} z(q^\omega; \lambda) \\ -y(q^\omega; \lambda) \end{pmatrix} \\
&= 0.
\end{aligned}$$

Because of the inequality (4.3), the right side of the equation (4.2) is identical zero only if $\bar{\lambda} = \lambda$, i.e., λ is real. \square

Theorem 4.5. *Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the positive eigenvalues of (4.1) and let $0 > \lambda_{-1} \geq \lambda_{-2} \geq \dots$ be the negative ones. Here it is assumed that each λ_j or λ_{-j} occurs in the sequences as given a number of times equal its multiplicity as a root of (4.1). Suppose \mathcal{H}_1 and \mathcal{H}_2 are two Hermitian matrix-valued functions of the class $\mathcal{P}(q^\omega)$ with $\mathcal{H}_1(t) \leq \mathcal{H}_2(t)$ for all $t \in [1, q^\omega] \cap \mathbb{T}$ and denote $\lambda_j(\mathcal{H})$ or $\lambda_{-j}(\mathcal{H})$ an eigenvalue depending on \mathcal{H} . Then $\lambda_j(\mathcal{H}_1) \geq \lambda_j(\mathcal{H}_2)$ and $\lambda_{-j}(\mathcal{H}_1) \leq \lambda_{-j}(\mathcal{H}_2)$ for all $j \in \mathbb{N}$.*

Proof. We shall only show that $\lambda_j(\mathcal{H}_1) \geq \lambda_j(\mathcal{H}_2)$ for all $j \in \mathbb{N}$. For the second result, its proof is as the proof of the first result. Let us consider the Hamiltonian boundary value problem

$$x_\varepsilon^\Delta(t) = \lambda_\varepsilon \mathcal{J} \mathcal{H}_\varepsilon(t) [\mathcal{M}^T \mathcal{M} x_\varepsilon^\sigma(t) + \mathcal{M} \mathcal{M}^T x_\varepsilon(t)], \quad x_\varepsilon(1) = q^\omega x_\varepsilon(q^\omega), \quad (4.4)$$

where $\mathcal{H}_\varepsilon(t) = \mathcal{H}_1(t) + \varepsilon(\mathcal{H}_2(t) - \mathcal{H}_1(t))$ for all $t \in [1, q^\omega] \cap \mathbb{T}$, and $0 \leq \varepsilon \leq 1$.

Assume $U_\varepsilon(t; \lambda_\varepsilon)$ is a fundamental matrix solution of (4.4). By (2.1) with $S = S_\varepsilon$, thus

$$U_\varepsilon(t; \lambda_\varepsilon) = e_{S_\varepsilon}(q^\omega, 1),$$

is a piecewise analytic function of the parameter ε , where

$$S_\varepsilon(t; \lambda_\varepsilon) = \lambda_\varepsilon (I - \mu(t) \lambda_\varepsilon \mathcal{J} \mathcal{H}_\varepsilon(t) \mathcal{M}^T \mathcal{M})^{-1} \mathcal{J} \mathcal{H}_\varepsilon(t).$$

If $\lambda_\varepsilon := \lambda_\varepsilon(\varepsilon) = \lambda_j(\mathcal{H}_\varepsilon)$ is a positive eigenvalue of (4.4), then it is also a piecewise analytic function of ε . Then we can choose a corresponding eigenvector η_ε with

$$x_\varepsilon(t; \lambda_\varepsilon) = U_\varepsilon(t; \lambda_\varepsilon) \eta_\varepsilon$$

subject to the normalization

$$-\int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^* \mathcal{J} \begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix}^{\Delta_t} \Delta t = \lambda_\varepsilon \int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^* \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix} \Delta t = 1, \quad (4.5)$$

where $x_\varepsilon(t; \lambda_\varepsilon) = \begin{pmatrix} y_\varepsilon(t; \lambda_\varepsilon) \\ z_\varepsilon(t; \lambda_\varepsilon) \end{pmatrix}$, or shortly $x_\varepsilon = \begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix}$. By differentiating the first integral of (4.5) with respect to ε ,

$$\int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^* \mathcal{J} \left(\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix}^{\Delta_t} \right)^{\Delta_\varepsilon} \Delta t + \int_1^{q^\omega} \left(\begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^* \right)^{\Delta_\varepsilon} \mathcal{J} \begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix}^{\Delta_t} \Delta t = 0, \quad (4.6)$$

where $\Delta_\varepsilon := \frac{\partial}{\partial \varepsilon}$. Also by differentiating the second integral of (4.5) with respect to ε ,

$$\begin{aligned} & \lambda_\varepsilon \int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^* (\mathcal{H}_2 - \mathcal{H}_1) \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix} \Delta t + \frac{d\lambda_\varepsilon}{d\varepsilon} \int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^* \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix} \Delta t \\ & + \lambda_\varepsilon \int_1^{q^\omega} \left\{ \left(\begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^* \right)^{\Delta_\varepsilon} \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix} + \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^* \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon \end{pmatrix}^{\Delta_\varepsilon} \right\} \Delta t = 0. \end{aligned} \quad (4.7)$$

Since $-\mathcal{J} \begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix}^{\Delta_t} = \lambda_\varepsilon \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}$ and by using the equation (4.6) together with the fact $\begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^{\Delta_\varepsilon} := \frac{\partial}{\partial \varepsilon} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix} \equiv \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}$, we obtain

$$\lambda_\varepsilon \int_1^{q^\omega} \left\{ \left(\begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^* \right)^{\Delta_\varepsilon} \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix} + \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^* \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^{\Delta_\varepsilon} \right\} \Delta t = 0.$$

Thus the equation (4.7) becomes

$$\frac{d\lambda_\varepsilon}{d\varepsilon} \int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^* \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix} \Delta t = -\lambda_\varepsilon \int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^* (\mathcal{H}_2 - \mathcal{H}_1) \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix} \Delta t,$$

but since from equation (4.5),

$$\int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^* \mathcal{H}_\varepsilon \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix} \Delta t = \frac{1}{\lambda_\varepsilon},$$

and $\mathcal{H}_1 \leq \mathcal{H}_2$, hence

$$\frac{1}{\lambda_\varepsilon} \frac{d\lambda_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon} (\ln \lambda_\varepsilon) = -\lambda_\varepsilon \int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^* (\mathcal{H}_2 - \mathcal{H}_1) \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix} \Delta t \leq 0,$$

i.e., the function λ_ε is nonincreasing. Now recall $\lambda_\varepsilon := \lambda_\varepsilon(\varepsilon) = \lambda_j(\mathcal{H}_\varepsilon)$, where $\mathcal{H}_\varepsilon = \mathcal{H}_1 + \varepsilon(\mathcal{H}_2 - \mathcal{H}_1)$. Obviously, since $\lambda_\varepsilon(0) = \lambda_j(\mathcal{H}_1)$ and $\lambda_\varepsilon(1) = \lambda_j(\mathcal{H}_2)$, $\lambda_j(\mathcal{H}_1) \geq \lambda_j(\mathcal{H}_2)$ for all $j \in \mathbb{N}$. The proof in the case where λ_ε is a positive eigenvalue is done. For the case where $\lambda_\varepsilon := \lambda_{-j}(\mathcal{H}_\varepsilon)$ is a negative eigenvalue, the following expression appears

$$\frac{d}{d\varepsilon} (\ln |\lambda_\varepsilon|) = |\lambda_\varepsilon| \int_1^{q^\omega} \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix}^* (\mathcal{H}_2 - \mathcal{H}_1) \begin{pmatrix} y_\varepsilon^\sigma \\ z_\varepsilon^\sigma \end{pmatrix} \Delta t \geq 0,$$

which implies λ_ε is nondecreasing. □

Theorem 4.6. *The multiplicity k_j of any eigenvalue λ_j of the equation (4.1) coincides with the number d_j of the linearly independent associated solutions of the equation (4.1).*

Proof. Let $V(\lambda) = U(q^\omega; \lambda) - q^{-\omega}I$ and λ_j be a root of $\det V(\lambda) = 0$. The number of the linearly independent solutions of (4.1) for $\lambda = \lambda_j$ is the number defect d_j of the matrix $V(\lambda)$. Because $V(\lambda)$ is rational matrix, the Smith–McMillan form can be applied for $V(\lambda)$. From Theorem 2.14, $V(\lambda) = \frac{P(\lambda)}{l(\lambda)}$, and the Smith–McMillan form for V is

$$V^{SM}(\lambda) = \text{diag} \left(\frac{\varepsilon_1(\lambda)}{\delta_1(\lambda)}, \frac{\varepsilon_2(\lambda)}{\delta_2(\lambda)}, \dots, \frac{\varepsilon_r(\lambda)}{\delta_r(\lambda)}, 0, \dots, 0 \right).$$

But $\det V(\lambda)$ is a nonzero rational function, thus also $\det V^{SM}(\lambda)$ is a nonzero rational function, furthermore, $\dim(V) = \text{rank}(P) = r$ and

$$\det V(\lambda) = C \det V^{SM} = C \frac{\varepsilon_1(\lambda)\varepsilon_2(\lambda) \dots \varepsilon_{\dim(V)}(\lambda)}{\delta_1(\lambda)\delta_2(\lambda) \dots \delta_{\dim(V)}(\lambda)}, \quad (4.8)$$

where C is a nonzero constant. If $(\lambda - \lambda_j) | \varepsilon_k(\lambda)$, then $\lambda - \lambda_j$ divides all polynomials $\varepsilon_p(\lambda)$ for all $p > k$. If the rank of $V(\lambda_j)$ is r_j and its defect is $d_j = \dim(V) - r_j$, then $\lambda - \lambda_j$ is a divisor of the last d_j polynomials $\varepsilon_k(\lambda)$ in (4.8). Because $\det V(\lambda)$ is a one-to-one rational function, this implies that $\varepsilon_i(\lambda)$ for $i \in \{1, 2, \dots, \dim(V)\}$, are simple polynomials. Hence the multiplicity k_j of λ_j is d_j . \square

5. STABILITY AND ANALYTIC PROPERTIES OF THE MULTIPLIERS

In this section, we shall discuss the strong stability for (4.1).

Definition 5.1. A point λ_0 is called a λ -point of stability of the Hamiltonian equation (4.1) if, for $\lambda = \lambda_0$, all solutions of (4.1) are bounded on the time scale \mathbb{T} . Furthermore, if, for $\lambda = \lambda_0$, all solutions of the equation of (4.1) having $\mathcal{H}(t)$ replaced $\tilde{\mathcal{H}}(t)$ which is ω -periodic and Hermitian and sufficiently close to $\mathcal{H}(t)$ (in some well-defined sense), are bounded on \mathbb{T} . Then we call $\lambda = \lambda_0$ a λ -point of strong stability of the Hamiltonian equation (4.1).

The following consequences follow from Theorem 3.8.

- (i) If we consider the Hamiltonian equation (4.1), we may obtain its neighborhoods that obey Theorem 3.8 by modifying the parameter λ .
- (ii) Since stability is expressed via the properties of the multipliers $\rho(\lambda)$ and strong stability, this means those properties of the multipliers $\rho(\lambda)$ are preserved with respect to the Hamiltonian perturbations. It is an important issue to discuss the multipliers with respect to these perturbations.

We have already shown that an eigenvalue λ of the boundary value problem (4.1) with $\mathcal{H} \in \mathcal{P}(q^\omega)$ is real. However, a complex eigenvalue λ of $(4.1)^*$ may occur and the following theorem shows that the multipliers depending on the complex eigenvalue λ of (4.1) are not on the unit circle.

Lemma 5.2. *If $\mathcal{H} \in \mathcal{P}(q^\omega)$, then the monodromy matrix of $(4.1)^*$ is \mathcal{J} -unitary, \mathcal{J} -increasing, or \mathcal{J} -decreasing depending on whether $\text{Im}\lambda$ is zero, positive, or negative.*

Proof. For any vector $\eta \in \mathbb{C}^n$, $\eta \neq 0$, the vector-valued function $x(t; \lambda) = U(t; \lambda)\eta$ is solution of the Hamiltonian equation $(4.1)^*$. With the given

$$x(t; \lambda) := \begin{pmatrix} y(t; \lambda) \\ z(t; \lambda) \end{pmatrix},$$

then by applying Lemma 4.2, we have

$$\begin{aligned} (U(q^\omega)\eta)^* \mathcal{J}U(q^\omega)\eta - (U(1)\eta)^* \mathcal{J}U(1)\eta \\ = -2i\text{Im}(\lambda) \int_1^{q^\omega} \begin{pmatrix} y^\sigma(t; \lambda) \\ z(t; \lambda) \end{pmatrix}^* \mathcal{H} \begin{pmatrix} y^\sigma(t; \lambda) \\ z(t; \lambda) \end{pmatrix} \Delta t, \end{aligned} \quad (5.1)$$

where i means the imaginary number. Multiplying both sides of (5.1) by the imaginary number i , we obtain

$$i\eta^* U^*(q^\omega) \mathcal{J}U(q^\omega)\eta - i\eta^* J\eta = 2\text{Im}(\lambda) \int_1^{q^\omega} \begin{pmatrix} y^\sigma(t; \lambda) \\ z(t; \lambda) \end{pmatrix}^* \mathcal{H} \begin{pmatrix} y^\sigma(t; \lambda) \\ z(t; \lambda) \end{pmatrix} \Delta t. \quad (5.2)$$

The left side of (5.2) is zero, positive, or negative depending on $\text{Im}(\lambda)$. This completes the proof. \square

Theorem 5.3. *Consider the Hamiltonian equation (4.1) with the complex eigenvalue λ , i.e., with $\text{Im}(\lambda) \neq 0$. Then half of the multipliers of (4.1) have moduli less than one and the other half have their moduli larger than one provided $\mathcal{H} \in \mathcal{P}(q^\omega)$.*

Proof. The proof is done by Lemma 5.2 and by Kreĭn [5, Theorem 1.1]. \square

Theorem 5.4. *The points of strong stability of (4.1) form an open set which is nonempty when (4.1) is of positive type, i.e., $\mathcal{H} \in \mathcal{P}(q^\omega)$.*

Proof. The proof goes as in [3] and [5] and also by applying Theorem 3.8 with $\lambda_0 \mathcal{H}$ as \mathcal{H} and $\lambda \mathcal{H}$ as $\tilde{\mathcal{H}}$, $\lambda \neq \lambda_0$. Thus if $\lambda_0 \in \mathbb{R}$ is a point of strong stability, then the set of strong stability points is open. \square

Let us consider the Hamiltonian equation (4.1). If it is stable, the monodromy matrix is of stable type, i.e., all multipliers are simple and have modulus one which may be first kind, second kind or mixed kind. The following are some interesting results.

- (i) If all multipliers are simple with multiplicity one and the stability is strong for any sufficiently small perturbation, the multipliers cannot leave the unit circle since they will break up the symmetry of multipliers.

- (ii) If there is a multiplier ρ_0 having its multiplicity of at least two, there may be taken away from the unit circle. In fact a newly appearing multiplier might be the multiplier of a perturbed Hamiltonian equation. A meeting of multipliers of the same kind will not move away from the unit circle, while the multipliers of different kinds that meet on the unit circle may move off the unit circle under a suitable perturbation.

6. REFERENCES

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IV. EXISTENCE OF PERIODIC SOLUTIONS OF A q -DIFFERENCE BOUNDARY VALUE PROBLEM

ABSTRACT

In this paper, we study a certain second-order q -difference equation subject to given boundary conditions. Using a recently introduced concept of periodic functions in quantum calculus, we establish the existence of solutions whose reciprocal square is periodic. The proof of our main result relies on an application of the Mountain Pass Theorem.

1. INTRODUCTION

Periodic solutions of difference (or differential) boundary value problems have been studied in many papers such as [3, 4, 7–9]. There are many approaches when seeking periodic solutions of difference (or differential) equations, such as critical point theory [5] (which includes minimax theory and Morse theory), fixed point theory, and many more. Throughout this paper, we consider the q -difference boundary value problem

$$\begin{aligned} x^{\Delta\Delta}(t) + \nabla F(qt, x(qt)) &= 0, \quad t \in \mathbb{T} := q^{\mathbb{N}_0} \\ x(1) &= q^{-\omega/2}x(q^\omega), \quad x^\Delta(1) = q^{\omega/2}x^\Delta(q^\omega), \end{aligned} \tag{1.1}$$

where

$$x^\Delta(t) = \frac{x(qt) - x(t)}{(q-1)t} \quad \text{for } t \in \mathbb{T},$$

$F : \mathbb{T} \times \mathbb{R}^m \rightarrow \mathbb{R}$ is continuously differentiable in the second variable and ω -periodic in the first variable, i.e., $F(t, u) = q^\omega F(q^\omega t, u)$ for all $(t, u) \in \mathbb{T} \times \mathbb{R}^m$, $\omega \in \mathbb{N}$, and $\nabla F(t, u)$ denotes the gradient of $F(t, u)$ in u .

In Section 3, we show that, by applying the Mountain Pass Theorem (Theorem 2.5), the problem (1.1) under certain hypotheses has at least one solution whose reciprocal square is ω -periodic. For the differential boundary value problem,

$$\begin{aligned} x''(t) + \nabla F(t, x(t)) &= 0, \quad t \in \mathbb{R}, \\ x(0) &= x(T), \quad x'(0) = x'(T), \end{aligned} \tag{1.2}$$

where $T > 0$, $F : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}$, the existence of T -periodic solutions under some hypotheses was established by Zhang and Zhou [9], while Long [4] obtained a similar result for the corresponding discrete boundary value problem. In both the continuous case and the discrete case, a nonnegative function is periodic if and only if its reciprocal square is periodic.

2. PRELIMINARIES AND AUXILIARY RESULTS

The following definitions and results are useful in order to prove the theorems in Section 3.

Definition 2.1 (Bohner and Peterson [2]). Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function. The expression

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}$$

is called the q -derivative of f .

Using the notation $f^\sigma(t) = f(qt)$, the q -derivatives of the product and quotient of $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are given by

$$(fg)\delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma$$

and

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}.$$

Definition 2.2 (Bohner and Peterson [2]). Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $s, t \in \mathbb{T}$ such that $s < t$.

Then

$$\int_s^t f(\xi) \Delta \xi := (q-1) \sum_{\tau \in [s, t) \cap \mathbb{T}} \tau f(\tau)$$

is called the *integral* on \mathbb{T} .

Definition 2.3 (Bohner and Chieochan [1]). A function $f : \mathbb{T} \rightarrow \mathbb{R}$ with

$$f(t) = q^\omega f(q^\omega t) \quad \text{for all } t \in \mathbb{T}$$

is called ω -periodic.

Let E be a real Banach space. $B_\rho(0)$ and $\partial B_\rho(0)$ denote the open ball centered at zero in E of radius ρ and the boundary of ball $B_\rho(0)$, respectively.

Definition 2.4. Let I be a continuously Fréchet differentiable functional defined on E . I is said to satisfy the Palais–Smale condition if any sequence $\{u_n\} \subset E$ for which $\{I(u_n)\}$ is bounded and $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ possesses a convergent subsequence in E .

Theorem 2.5 (Mountain Pass Theorem [6]). *Let $J \in \mathcal{C}^1(E, \mathbb{R})$. Suppose J satisfies the Palais–Smale condition, $J(0) = 0$,*

(J_1) there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho(0)} \geq \alpha$, and

(J_2) there is an $e \in E \setminus \partial B_\rho(0)$ such that $J(e) \leq 0$.

Then J possesses a critical value $c \geq \alpha$ which can be characterized by

$$c = \inf_{g \in \Gamma} \max_{u \in g[0,1]} J(u),$$

where $\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}$.

To prove the main theorems in Section 3, we introduce a functional for the problem (1.1) in the following way. Let

$$S = \{x = \{x(t)\} : x(t) \in \mathbb{R}^m, t \in \mathbb{T} \cup \{1/q\}\}.$$

and define the vector subspace of S

$$E_\omega = \{x = \{x(t)\} \in S : x(t) = q^{-\omega/2} x(q^\omega t), t \in \mathbb{T} \cup \{1/q\}\}.$$

Now E_ω can be equipped with the norm $\|\cdot\|_{E_\omega}$ and the inner product $\langle \cdot, \cdot \rangle_{E_\omega}$ for any $x, y \in E_\omega$ by

$$\|x\|_{E_\omega} := \left(\sum_{t \in Q_\omega} |x(t)|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \langle x, y \rangle_{E_\omega} := \sum_{t \in Q_\omega} x(t) \cdot y(t),$$

where

$$Q_\omega = \{q^k : 1 \leq k \leq \omega - 1\},$$

$|\cdot|$ denotes the usual norm in \mathbb{R}^m , and $x(t) \cdot y(t)$ denotes the usual scalar product in \mathbb{R}^m . It is simple to show that E_ω is isomorphic to $\mathbb{R}^{\omega m}$, and moreover, $(E_\omega, \langle \cdot, \cdot \rangle_{E_\omega})$ is a Hilbert space. For any given number $r > 1$, we let

$$\|x\|_r = \left(\sum_{t \in Q_\omega} |x(t)|^r \right)^{\frac{1}{r}}$$

for all $x \in E_\omega$. By Hölder's inequality, $\|\cdot\|_r$ is a norm on E_ω . Thus we have $\|\cdot\|_{E_\omega} = \|\cdot\|_2$. Then there exist some constants C_1 and C_2 such that $0 < C_1 \leq C_2$ and

$$C_1 \|x\|_r \leq \|x\|_2 \leq C_2 \|x\|_r \quad \text{for all } x \in E_\omega. \quad (2.1)$$

Furthermore,

$$\|x\|_1 \leq \sqrt{\omega} \|x\|_2 \quad \text{for all } x \in E_\omega. \quad (2.2)$$

Now the functional J on E_ω is defined by

$$J(x) = \int_1^{q^\omega} \left(-\frac{1}{2} |x^\Delta(t)|^2 + F(qt, x(qt)) \right) \Delta t \quad \text{for all } x \in E_\omega. \quad (2.3)$$

By Definition 2.1 and 2.2, the functional J can be rewritten as

$$J(x) = \sum_{t \in Q_\omega} \left\{ -\frac{1}{2(q-1)t} [x^2(t) - 2x(t)x(qt) + x^2(qt)] + (q-1)tF(qt, x(qt)) \right\} \quad (2.4)$$

for all $x \in E_\omega$. Suppose

$$\nabla F(t, x) = f(t, x) \in C(\mathbb{T} \times \mathbb{R}^m, \mathbb{R}^m), \quad \text{where } f = (f_1, f_2, \dots, f_m)^T.$$

Let

$$x = \{x(t)\} \in E_\omega, \quad \text{where } x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T,$$

and denote $\mu(t) := (q-1)t$ for all $t \in \mathbb{T} \cup \{1/q\}$. By the assumption, $F(t, x) = q^\omega F(q^\omega t, x)$ for all $(t, x) \in \mathbb{T} \times \mathbb{R}^m$,

$$\frac{\partial J(x)}{\partial x_j(t)} = \mu(t/q) [x_j^{\Delta\Delta}(t/q) + f_j(t, x_j(t))]$$

for all $t \in Q_\omega$ and $j \in \{1, 2, \dots, m\}$. Therefore, $x = \{x(t)\} \in E_\omega$ is a critical point of J , that is, $J'(x) = 0$ if and only if for each $j \in \{1, 2, \dots, m\}$,

$$x_j^{\Delta\Delta}(t/q) + f_j(t, x_j(t)) = 0$$

i.e., $x^{\Delta\Delta}(t) + \nabla F(qt, x(qt)) = 0$ for all $t \in Q_\omega$. Hence, if $x \in E_\omega$ is a critical point of J , then it is a solution of (1.1), and the reciprocal square of x , i.e., $1/x^2$, is ω -periodic. Let

$$z = (z(1)^T, z(q)^T, \dots, z(q^{\omega-1})^T)T \quad \text{with} \quad z(t) = (z_1(t), z_2(t), \dots, z_m(t))^T \in \mathbb{R}^m$$

for all $t \in Q_\omega$. We have

$$\begin{aligned} Pz &= (z_1(1), z_1(q), \dots, z_1(q^{\omega-1}), z_2(1), z_2(q), \dots, z_2(q^{\omega-1}), \dots \\ &\quad \dots, z_m(1), z_m(q), \dots, z_m(q^{\omega-1}))^T, \end{aligned}$$

where P is the $\omega m \times \omega m$ -matrix given by

$$P = \begin{pmatrix}
& & & m & & & & 2m & & & & & & (\omega-1)m+1 \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0\dots & 0\dots & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 1 & 0 & \dots & 0\dots & 0\dots & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0\dots & 0\dots & 1 & 0 & \dots & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & \dots & 0\dots & 0\dots & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & \dots & 0\dots & 0\dots & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0\dots & 0\dots & 0 & 1 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 1 & 0 & 0 & \dots & 0\dots & 0\dots & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 1\dots & 0\dots & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0\dots & 0\dots & 0 & 0 & \dots & 1
\end{pmatrix} \begin{matrix} \\ \\ \\ \omega \\ \\ \\ 2\omega \\ \\ (m-1)\omega+1 \\ \\ \end{matrix}$$

Then the functional J given by (2.4) can be rewritten as

$$J(z) = -\frac{1}{2}\langle APz, Pz \rangle + \sum_{t \in Q_\omega} \mu(t)F(qt, z(qt)) \quad \text{for all } z \in E_\omega, \quad (2.5)$$

where

$$A = \begin{pmatrix} B & & 0 \\ & B & \\ & & \ddots \\ 0 & & & B \end{pmatrix}_{\omega m \times \omega m}$$

and

$$B = \frac{1}{q-1} \begin{pmatrix} [q]_0 & -1 & 0 & 0 & \dots & 0 & -q^{-\omega/2+1} \\ -1 & [q]_1 & -\frac{1}{q} & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{q} & [q]_2 & -\frac{1}{q^2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & -\frac{1}{q^{\omega-3}} & [q]_{\omega-2} & -\frac{1}{q^{\omega-2}} \\ -q^{-\omega/2+1} & 0 & 0 & 0 & \dots & -\frac{1}{q^{\omega-2}} & [q]_{\omega-1} \end{pmatrix}_{\omega \times \omega},$$

and $[q]_0 = 1+q$, $[q]_n = \frac{1}{q^{n-1}} + \frac{1}{q^n}$, $n \in \{1, 2, \dots, \omega-1\}$. Let $D = P^{-1}AP$. Since $P^{-1} = P^T$, $D^T = D$ and

$$J(z) = -\frac{1}{2} \langle Dz, z \rangle + \sum_{t \in Q_\omega} \mu(t) F(qt, z(qt)) \quad \text{for all } z \in E_\omega. \quad (2.6)$$

By matrix theory, the matrices A and D have the same real eigenvalues with the same multiplicities. It is simple to show that the matrix B is positive definite, i.e., all eigenvalues of B are positive real numbers. This implies that each eigenvalue of the matrix B is also an eigenvalue of the matrix D with multiplicity m .

3. MAIN RESULTS

Throughout this section, we denote by λ_{\min} and λ_{\max} the minimum and maximum eigenvalues of the matrix D given in (2.6), respectively, and let

$$\nabla F(t, x) = f(t, x) \in C(\mathbb{T} \times \mathbb{R}^m, \mathbb{R}^m).$$

We apply the Mountain Pass Theorem to prove the main theorems in this section.

Theorem 3.1. *Suppose that $F(t, z)$ satisfies the following:*

(H₁) *there exists $\omega \in \mathbb{N}$ such that $F(t, z) = q^\omega F(q^\omega t, z)$ for any $(t, z) \in \mathbb{T} \times \mathbb{R}^m$;*

(H₂) *there is a constant M_0 such that $|f(t, z)| \leq M_0$ for all $(t, z) \in \mathbb{T} \times \mathbb{R}^m$;*

(H₃) *$F(t, z) \rightarrow \infty$ uniformly for all $t \in \mathbb{T}$ as $|z| \rightarrow \infty$.*

Then the problem (1.1) has at least one solution.

Lemma 3.2. *Under the hypotheses of Theorem 3.1, the functional J satisfies the Palais–Smale condition.*

Proof. Suppose that $\{x^{(k)}\} \subset E_\omega$ is such that for all $k \in \mathbb{N}$, $|J(x^{(k)})| \leq M_2$ for some $M_2 > 0$, and $J'(x^{(k)}) \rightarrow 0$ as $k \rightarrow \infty$. Then, for the sufficiently large k ,

$$\langle J'(x^{(k)}), x \rangle \geq -\|x\|_2.$$

We have

$$\begin{aligned} -\|x^{(k)}\|_2 &\leq \langle J'(x^{(k)}), x^{(k)} \rangle \\ &= -\langle Dx^{(k)}, x^{(k)} \rangle + \sum_{t \in Q_\omega} \mu(t) f(qt, x^{(k)}(qt)) x^{(k)}(qt) \\ &\stackrel{(H_2)}{\leq} -\lambda_{\min} \|x^{(k)}\|_2^2 + M_0(q-1)q^{\omega-1} \sum_{t \in Q_\omega} |x^{(k)}(qt)| \\ &= -\lambda_{\min} \|x^{(k)}\|_2^2 + M_0(q-1)q^{\omega-1} \sum_{t \in Q_\omega \setminus \{q^{\omega-1}\}} |x^{(k)}(qt)| + q^{\frac{\omega}{2}} |x^{(k)}(1)| \end{aligned}$$

$$\begin{aligned}
&\leq -\lambda_{\min}\|x^{(k)}\|_2^2 + M_0(q-1)q^{\frac{3\omega}{2}-1} \sum_{t \in Q_\omega} |x^{(k)}(t)| \\
&= -\lambda_{\min}\|x^{(k)}\|_2^2 + M_0(q-1)q^{\frac{3\omega}{2}-1} \|x^{(k)}\|_1 \\
&\stackrel{(2.2)}{\leq} -\lambda_{\min}\|x^{(k)}\|_2^2 + M_0(q-1)q^{\frac{3\omega}{2}-1} \sqrt{\omega} \|x^{(k)}\|_2.
\end{aligned}$$

This gives

$$\|x^{(k)}\|_2 \leq \frac{1}{\lambda_{\min}} \left(1 + M_0(q-1)q^{\frac{3\omega}{2}-1} \sqrt{\omega} \right)$$

for all $k \in \mathbb{N}$, i.e., $\{x^{(k)}\}$ is bounded for all $k \in \mathbb{N}$. Since E_ω is finite dimensional, there exists a convergent subsequence of $\{x^{(k)}\}$. Hence J satisfies the Palais–Smale condition. \square

Proof of Theorem 3.1. By Lemma 3.2, the functional J satisfies the Palais–Smale condition. By hypothesis (H₃), there exist $\rho > 0$ and $R > 0$ such that

$$\rho^2 \leq \frac{2\omega}{4\lambda_{\max}}(q-1)R \quad \text{and} \quad F(t, z) \geq R \quad \text{for all} \quad (t, z) \in \mathbb{T} \times \partial B_\rho(0) \cap E_\omega.$$

Thus, for any $z \in \partial B_\rho(0) \cap E_\omega$, we have

$$\begin{aligned}
J(z) &= -\frac{1}{2} \langle Dz, z \rangle + \sum_{t \in Q_\omega} \mu(t) F(qt, z(qt)) \\
&\geq -\frac{\lambda_{\max}}{2} \|z\|_2^2 + \omega(q-1)R \\
&= -\frac{\lambda_{\max}}{2} \rho^2 + \omega(q-1)R \\
&\geq \frac{3\omega}{4} (q-1)R.
\end{aligned}$$

Hence condition (J₁) of the Mountain Pass Theorem holds. Because of $\nabla F(t, z) = f(t, z)$ and the hypothesis (H₂),

$$|F(t, z)| \leq M_1 + M_0|z| \quad \text{for all} \quad (t, z) \in \mathbb{T} \times \mathbb{R}^m$$

and for some number $M_1 > 0$. Let $y \in E_\omega$ be arbitrary. Then

$$\begin{aligned}
J(y) &= -\frac{1}{2}\langle Dy, y \rangle + \sum_{t \in Q_\omega} \mu(t) F(qt, y(qt)) \\
&\leq -\frac{\lambda_{\min}}{2} \|y\|_2^2 + \sum_{t \in Q_\omega} \mu(t) |F(qt, y(qt))| \\
&\leq -\frac{\lambda_{\min}}{2} \|y\|_2^2 + \sum_{t \in Q_\omega} \mu(t) (M_1 + M_0 |y(qt)|) \\
&\leq -\frac{\lambda_{\min}}{2} \|y\|_2^2 + M_1(q^\omega - 1) + M_0 q^{\omega-1} (q-1) \sum_{t \in Q_\omega} |y(qt)| \\
&\leq -\frac{\lambda_{\min}}{2} \|y\|_2^2 + M_1(q^\omega - 1) + M_0 q^{\frac{5\omega}{2}-1} (q-1) \|y\|_1 \\
&\stackrel{(2.2)}{\leq} -\frac{\lambda_{\min}}{2} \|y\|_2^2 + M_1(q^\omega - 1) + M_0 q^{\frac{5\omega}{2}-1} (q-1) \sqrt{\omega} \|y\|_2 \\
&= \|y\|_2^2 \left(-\frac{\lambda_{\min}}{2} + M_0 q^{\frac{5\omega}{2}-1} (q-1) \frac{\sqrt{\omega}}{\|y\|_2} \right) + M_1(q^\omega - 1) \rightarrow -\infty
\end{aligned}$$

as $\|y\|_2 \rightarrow \infty$. Then there exists a sequence $e \in E_\omega \setminus \partial B_\rho(0)$ such that $\|e\|_2$ is sufficiently large and

$$J(e) \leq \|e\|_2^2 \left(-\frac{\lambda_{\min}}{2} + M_0 q^{\frac{5\omega}{2}-1} (q-1) \frac{\sqrt{\omega}}{\|e\|_2} \right) + M_1(q^\omega - 1) \leq 0.$$

Thus J satisfies condition (J_2) of the Mountain Pass Theorem. Hence the proof is complete. \square

Theorem 3.3. *Suppose that $F(t, z)$ satisfies (H_1) and*

(H_4) there exist constants R_1 and $\alpha \in (1, 2)$ such that $0 < z f(t, z) \leq \alpha F(t, z)$ for all $(t, z) \in \mathbb{T} \times \mathbb{R}^m$, $|z| \geq R_1$;

(H_5) there exist constants $\beta_1, \beta_2 > 0$ and $\gamma > 2$ such that $F(t, z) \geq a_1(t)|z|^\gamma - a_2(t)$ for all $(t, z) \in \mathbb{T} \times \mathbb{R}^m$, where the functions $a_1, a_2 : \mathbb{T} \rightarrow \mathbb{R}^+$ are given by $a_1(t) = \frac{\beta_1}{t}$ and $a_2(t) = \frac{\beta_2}{t}$;

Then the problem (1.1) has at least one solution.

Lemma 3.4. *Under the hypotheses of Theorem 3.3, the functional J satisfies the Palais–Smale condition.*

Proof. Assume $\{x^{(k)}\} \subset E_\omega$ for all $k \in \mathbb{N}$ such that $|J(x^{(k)})| \leq M_5$ for some $M_5 > 0$ and $J'(x^{(k)}) \rightarrow 0$ as $k \rightarrow \infty$. Since $\lim_{k \rightarrow \infty} J'(x^{(k)}) = 0$, for sufficiently large k , we have

$$-\frac{1}{2}\|x^{(k)}\|_2 \leq -\frac{1}{2}\langle J'(x^{(k)}), x^{(k)} \rangle \leq \frac{1}{2}\|x^{(k)}\|_2.$$

Then

$$\begin{aligned} M_5 + \frac{1}{2}\|x^{(k)}\|_2 &\geq J(x^{(k)}) - \frac{1}{2}\langle J'(x^{(k)}), x^{(k)} \rangle \\ &= \sum_{t \in Q_\omega} \mu(t) \left(F(qt, x^{(k)}(qt)) - \frac{1}{2}f(qt, x^{(k)}(qt))x^{(k)}(qt) \right). \end{aligned}$$

Let

$$\mathcal{A}_1 := \{t \in Q_\omega : |x^{(k)}(qt)| \geq R_1\} \quad \text{and} \quad \mathcal{A}_2 := \{t \in Q_\omega : |x^{(k)}(qt)| < R_1\},$$

where the constant number R_1 is from (H_4) . Then

$$\begin{aligned} &\sum_{t \in Q_\omega} \mu(t) \left(F(qt, x^{(k)}(qt)) - \frac{1}{2}f(qt, x^{(k)}(qt))x^{(k)}(qt) \right) \\ &= \sum_{t \in Q_\omega} \mu(t)F(qt, x^{(k)}(qt)) - \frac{1}{2} \sum_{t \in \mathcal{A}_1} \mu(t)f(qt, x^{(k)}(qt))x^{(k)}(qt) \\ &\quad - \frac{1}{2} \sum_{t \in \mathcal{A}_2} \mu(t)f(qt, x^{(k)}(qt))x^{(k)}(qt) \\ &\stackrel{(H_4)}{\geq} \sum_{t \in Q_\omega} \mu(t)F(qt, x^{(k)}(qt)) - \frac{\alpha}{2} \sum_{t \in \mathcal{A}_1} \mu(t)F(qt, x^{(k)}(qt)) \\ &\quad - \frac{1}{2} \sum_{t \in \mathcal{A}_2} \mu(t)f(qt, x^{(k)}(qt))x^{(k)}(qt) \\ &= \left(1 - \frac{\alpha}{2}\right) \sum_{t \in Q_\omega} \mu(t)F(qt, x^{(k)}(qt)) \\ &\quad + \frac{1}{2} \sum_{t \in \mathcal{A}_2} \mu(t) [\alpha F(qt, x^{(k)}(qt)) - f(qt, x^{(k)}(qt))x^{(k)}(qt)]. \end{aligned}$$

Since $\alpha F(t, z) - f(t, z)z$ is continuous with respect to (t, z) , there exists a constant number $M_6 > 0$ such that $|z| < R_1$, and then $\alpha F(t, z) - f(t, z)z \geq -M_6$ for all $t \in Q_\omega$. Therefore,

$$\begin{aligned}
M_5 + \frac{1}{2}\|x^{(k)}\|_2 &\geq \left(1 - \frac{\alpha}{2}\right) \sum_{t \in Q_\omega} \mu(t) F(qt, x^{(k)}(qt)) - \frac{1}{2}M_6(q^\omega - 1) \\
&\stackrel{(H_5)}{\geq} \left(1 - \frac{\alpha}{2}\right) \sum_{t \in Q_\omega} \mu(t) [a_1(qt)|x^{(k)}(qt)|^\gamma - a_2(qt)] - \frac{1}{2}M_6(q^\omega - 1) \\
&\geq \left(1 - \frac{\alpha}{2}\right) \frac{\beta_1}{q}(q-1) \sum_{t \in Q_\omega} |x^{(k)}(qt)|^\gamma - (q-1)\omega \frac{\beta_2}{q} \left(1 - \frac{\alpha}{2}\right) - \frac{1}{2}M_6(q^\omega - 1) \\
&\geq \left(1 - \frac{\alpha}{2}\right) \frac{\beta_1}{q}(q-1)\|x^{(k)}\|_\gamma^\gamma - (q-1)\omega \frac{\beta_2}{q} \left(1 - \frac{\alpha}{2}\right) - \frac{1}{2}M_6(q^\omega - 1) \\
&\geq \left(1 - \frac{\alpha}{2}\right) \frac{\beta_1}{q}(q-1)\|x^{(k)}\|_\gamma^\gamma - M_7,
\end{aligned}$$

where

$$M_7 = (q-1)\omega \frac{\beta_2}{q} \left(1 - \frac{\alpha}{2}\right) + \frac{1}{2}M_6(q^\omega - 1).$$

By the inequality (2.1), we have

$$M_5 + \frac{1}{2}\|x^{(k)}\|_2 \geq \left(1 - \frac{\alpha}{2}\right) \frac{\beta_1}{q}(q-1) \frac{\|x^{(k)}\|_2^\gamma}{C_2^\gamma} - M_7.$$

Hence

$$\left(1 - \frac{\alpha}{2}\right) \frac{\beta_1}{q}(q-1) \frac{\|x^{(k)}\|_2^\gamma}{C_2^\gamma} - \frac{1}{2}\|x^{(k)}\|_2 \leq M_5 + M_7.$$

This implies that $\{x^{(k)}\}$ is bounded for all $k \in \mathbb{N}$ because $2 < \gamma < \infty$. Hence J satisfies the Palais–Smale condition. \square

Proof of Theorem 3.3. By Lemma 3.4, J satisfies Palais–Smale condition. Let y be any element in E_ω . Then we have

$$\begin{aligned}
J(y) &= -\frac{1}{2}\langle Dy, y \rangle + \sum_{t \in Q_\omega} \mu(t) F(qt, y(qt)) \\
&\stackrel{(H_5)}{\geq} -\frac{\lambda_{\max}}{2}\|y\|_2^2 + \sum_{t \in Q_\omega} \mu(t) [a_1(qt)|y(qt)|^\gamma - a_2(qt)] \\
&\geq -\frac{\lambda_{\max}}{2}\|y\|_2^2 + \frac{\beta_1}{q}(q-1)\|y\|_\gamma^\gamma - \frac{\beta_2}{q}\omega(q-1)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(2.1)}{\geq} -\frac{\lambda_{\max}}{2}\|y\|_2^2 + \frac{\beta_1}{q}(q-1)\frac{\|y\|_2^\gamma}{C_2^\gamma} - \frac{\beta_2}{q}\omega(q-1) \\
&= \|y\|_2^\gamma \left[\frac{\beta_1}{q}(q-1)\frac{1}{C_2^\gamma} - \frac{\lambda_{\max}}{2\|y\|_2^{\gamma-2}} \right] - \frac{\beta_2}{q}\omega(q-1).
\end{aligned}$$

Since $J(y) \rightarrow \infty$ as $\|y\|_2 \rightarrow \infty$, there exists $\rho > 0$ sufficiently large such that for any $z \in \mathbb{T} \cap \partial B_\rho(0)$,

$$J(z) \geq \rho^\gamma \left[\frac{\beta_1}{q}(q-1)\frac{1}{C_2^\gamma} - \frac{\lambda_{\max}}{2\rho^{\gamma-2}} \right] - \frac{\beta_2}{q}\omega(q-1) > 0.$$

Hence (J_1) of the Mountain Pass Theorem holds. Next we prove that J satisfies (J_2) of the Mountain Pass Theorem. By integrating both sides of the inequality $zf(t, z) \leq \alpha F(t, z)$ given by (H_4) for any $(t, z) \in \mathbb{T} \times \mathbb{R}^m$ such that $|z| \geq R_1 > 0$, we have $F(t, z) \leq b_1|z|^\alpha + b_2$ for some constants $b_1, b_2 > 0$. Let $x \in E_\omega$ be arbitrary. Then

$$\begin{aligned}
J(x) &= -\frac{1}{2}\langle Dx, x \rangle + \sum_{t \in Q_\omega} \mu(t)F(qt, x(qt)) \\
&\leq -\frac{\lambda_{\min}}{2}\|x\|_2^2 + b_1 \sum_{t \in Q_\omega} \mu(t)|x(qt)|^\alpha + b_2(q^\omega - 1) \\
&\leq -\frac{\lambda_{\min}}{2}\|x\|_2^2 + b_1(q-1)q^{\omega-1} \left[\sum_{t \in Q_\omega \setminus \{q^{\omega-1}\}} |x(qt)|^\alpha + q^{\frac{\alpha\omega}{2}} |x(1)|^\alpha \right] \\
&\quad + b_2(q^\omega - 1) \\
&\leq -\frac{\lambda_{\min}}{2}\|x\|_2^2 + b_1(q-1)q^{\omega(1+\frac{\alpha}{2})-1} \sum_{t \in Q_\omega} |x(t)|^\alpha + b_2(q^\omega - 1) \\
&= -\frac{\lambda_{\min}}{2}\|x\|_2^2 + b_1(q-1)q^{\omega(1+\frac{\alpha}{2})-1}\|x\|_\alpha^\alpha + b_2(q^\omega - 1) \\
&\stackrel{(2.1)}{\leq} -\frac{\lambda_{\min}}{2}\|x\|_2^2 + b_1(q-1)q^{\omega(1+\frac{\alpha}{2})-1}\frac{\|x\|_2^\alpha}{C_1^\alpha} + b_2(q^\omega - 1).
\end{aligned}$$

It follows that $J(x) \rightarrow -\infty$ as $\|x\|_2 \rightarrow \infty$. Then there exists a sequence $e \in E_\omega \setminus \partial B_\rho(0)$ such that $J(e) \leq 0$. So condition (J_2) of the Mountain Pass Theorem holds. The proof is complete. \square

Theorem 3.5. *Suppose that $F(t, z)$ satisfies (H_1) and*

(H₆) there exists a constant $\alpha \in (1, 2)$ such that $0 < zf(t, z) \leq \alpha F(t, z)$ for all $(t, z) \in \mathbb{T} \times \mathbb{R}^m$, with $|z| \neq 0$;

(H₇) there exist constants $\beta > 0$ and $\gamma \in (1, \alpha]$ such that $F(t, z) \geq a(t)|z|^\gamma$ for all $(t, z) \in \mathbb{T} \times \mathbb{R}^m$, where the function $a : \mathbb{T} \rightarrow \mathbb{R}^+$ is given by $a(t) = \frac{\beta}{t}$.

Then the problem (1.1) has at least one solution.

Proof. Under the given assumptions, we can show as in the proof of Lemma 3.4 that J satisfies Palais–Smale condition. Moreover, for any $x \in E_\omega$, we have

$$\begin{aligned}
 J(x) &= -\frac{1}{2}\langle Dx, x \rangle + \sum_{t \in Q_\omega} \mu(t) F(qt, x(qt)) \\
 &\stackrel{(H_7)}{\geq} -\frac{\lambda_{\max}}{2} \|x\|_2^2 + \sum_{t \in Q_\omega} \mu(t) a(qt) |x(qt)|^\gamma \\
 &\geq -\frac{\lambda_{\max}}{2} \|x\|_2^2 + \frac{\beta}{q} (q-1) \|x\|_\gamma^\gamma \\
 &\stackrel{(2.1)}{\geq} -\frac{\lambda_{\max}}{2} \|x\|_2^2 + \frac{\beta}{q} (q-1) \frac{\|x\|_2^\gamma}{C_2^\gamma}.
 \end{aligned}$$

Since there is a real number $\rho > 0$ such that $\rho^{2-\gamma} < \frac{\beta(q-1)}{2q\lambda_{\max}C_2^\gamma}$, for all $y \in \mathbb{T} \cap \partial B_\rho(0)$, we have

$$J(y) \geq -\frac{\beta(q-1)}{4qC_2^\gamma} \rho^\gamma + \frac{\beta(q-1)}{qC_2^\gamma} \rho^\gamma = \frac{3\beta(q-1)}{4qC_2^\gamma} \rho^\gamma > 0.$$

So condition (J₁) of the Mountain Pass Theorem holds. Next we show that J satisfies (J₂) of the Mountain Pass Theorem. By integrating the inequality $zf(t, z) \leq \alpha F(t, z)$ given by (H₆), $F(t, z) \leq b_3|z|^\alpha + b_4$ for some constants $b_3, b_4 > 0$. Then for any $y \in E_\omega$, we have

$$\begin{aligned}
 J(y) &= -\frac{1}{2}\langle Dy, y \rangle + \sum_{t \in Q_\omega} \mu(t) F(qt, y(qt)) \\
 &\leq -\frac{\lambda_{\min}}{2} \|y\|_2^2 + \sum_{t \in Q_\omega} \mu(t) [b_3|y(qt)|^\alpha + b_4] \\
 &\leq -\frac{\lambda_{\min}}{2} \|y\|_2^2 + b_3(q-1)q^{\omega-1} \sum_{t \in Q_\omega \setminus \{q^{\omega-1}\}} |y(qt)|^\alpha + q^{\frac{\alpha\omega}{2}} |y(1)|^\alpha + b_4(q^\omega - 1) \\
 &\leq -\frac{\lambda_{\min}}{2} \|y\|_2^2 + b_3(q-1)q^{\omega(1+\frac{\alpha}{2})-1} \sum_{t \in Q_\omega} |y(t)|^\alpha + b_4(q^\omega - 1)
 \end{aligned}$$

$$\leq -\frac{\lambda_{\min}}{2}\|y\|_2^2 + b_3(q-1)q^{\omega(1+\frac{\alpha}{2})-1}\frac{\|y\|_2^\alpha}{C_1^\alpha} + b_4(q^\omega - 1) \rightarrow -\infty$$

as $\|y\|_2 \rightarrow \infty$. It follows that there exist a real number $\rho > 0$ and a sequence $e \in E_\omega \cap \partial B_\rho(0)$ such that if $\|e\|_2$ is sufficiently large, then $J(e) \leq 0$. Thus condition (J_2) of the Mountain Pass Theorem holds. \square

Finally, we give an example illustrating Theorem 3.5.

Example 3.6. Let us consider the q -difference boundary value problem

$$\begin{aligned} z^{\Delta\Delta}(t) + \frac{a}{qt}(\beta_1 + 2)z(qt)|z(qt)|^{\beta_1} + \frac{b}{qt}(\beta_2 + 2)z(qt)|z(qt)|^{\beta_2} &= 0, \quad t \in \mathbb{T}, \\ z(1) &= q^{-\omega/2}z(q^\omega), \quad z^\Delta(1) = q^{\omega/2}z^\Delta(q^\omega), \end{aligned} \quad (3.1)$$

where $a > 0$, $b \geq 0$, and $-1 < \beta_1 \leq \beta_2 < 0$. Then we have

$$\nabla F(t, z) = \frac{a}{t}(\beta_1 + 2)z|z|^{\beta_1} + \frac{b}{t}(\beta_2 + 2)z|z|^{\beta_2}$$

and

$$F(t, z) = \frac{a}{t}|z|^{\beta_1+2} + \frac{b}{t}|z|^{\beta_2+2}$$

for all $(t, z) \in \mathbb{T} \times \mathbb{R}^m$ and some $m \in \mathbb{N}$. It is clear that F satisfies (H_1) . Let $\alpha = \beta_2 + 2$ and $\gamma = \beta_1 + 2$. It is simple to check that the assumptions (H_6) and (H_7) of Theorem 3.5 hold. Hence, for any given $\omega \in \mathbb{N}$, the problem (3.1) has at least one solution z , and then the reciprocal square of the solution z , i.e., $1/z^2$, is ω -periodic.

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V. POSITIVE PERIODIC SOLUTIONS OF HIGHER-ORDER FUNCTIONAL q -DIFFERENCE EQUATIONS

ABSTRACT

In this paper, using the recently introduced concept of periodic functions in quantum calculus, we study the existence of positive periodic solutions of a certain higher-order functional q -difference equation. Just as for the well-known continuous and discrete versions, we use a fixed point theorem in a cone in order to establish the existence of a positive periodic solution.

1. INTRODUCTION

The existence of positive periodic solutions of functional difference equations has been studied by many authors such as Zhang and Cheng [2], Zhu and Li [5], and Wang and Luo [6]. Some well-known models which are first-order functional difference equations are, for example (see [6]),

(i) the discrete model of blood cell production:

$$\begin{aligned}\Delta x(n) &= -a(n)x(n) + b(n)\frac{1}{1 + x^k(n - \tau(n))}, & k \in \mathbb{N}, \\ \Delta x(n) &= -a(n)x(n) + b(n)\frac{x(n - \tau(n))}{1 + x^k(n - \tau(n))}, & k \in \mathbb{N},\end{aligned}$$

(ii) the periodic Michaelis–Menton model:

$$\Delta x(n) = a(n)x(n) \left[1 - \sum_{j=1}^k \frac{a_j(n)x(n - \tau_j(n))}{1 + c_j(n)x(n - \tau_j(n))} \right], \quad k \in \mathbb{N},$$

(iii) the single species discrete periodic population model:

$$\Delta x(n) = x(n) \left[a(n) - \sum_{j=1}^k b_j(n)x(n - \tau_j(n)) \right], \quad k \in \mathbb{N}.$$

This paper studies the existence of periodic solutions of the m -order functional q -difference equations

$$x(q^m t) = a(t)x(t) + f(t, x(t/\tau(t))), \quad (1.1)$$

$$x(q^m t) = a(t)x(t) - f(t, x(t/\tau(t))), \quad (1.2)$$

where $a : q^{\mathbb{N}_0} \rightarrow [0, \infty)$ with $a(t) = a(q^\omega t)$, $f : q^{\mathbb{N}_0} \times \mathbb{R} \rightarrow [0, \infty)$ is continuous and ω -periodic, i.e., $f(t, u) = q^\omega f(q^\omega t, u)$, and $\tau : q^{\mathbb{N}_0} \rightarrow q^{\mathbb{N}_0}$ satisfies $t \geq \tau(t)$ for all $t \in q^{\mathbb{N}_0}$. A few examples of the function a are given by $a(t) = c$, where c is constant for any $t \in q^{\mathbb{N}_0}$, and $a(t) = d_t$, where d_t are constants assigned for each $t \in \{q^k : 0 \leq k \leq \omega - 1\}$. By

applying the fixed point theorem (Theorem 1.2) in a cone, we will prove later that (1.1) and (1.2) have positive periodic solutions. The definition of periodic functions on the so-called q -time scale $q^{\mathbb{N}_0}$ has recently been given by the authors [1] as follows.

Definition 1.1 (Bohner and Chieochan [1]). A function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ satisfying

$$f(t) = q^\omega f(q^\omega t) \quad \text{for all } t \in q^{\mathbb{N}_0}$$

is called ω -periodic.

Theorem 1.2 (Fixed point theorem in a cone [3,4]). *Let X be a Banach space and P be a cone in X . Suppose Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and suppose that $\Phi : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow P$ is a completely continuous operator such that*

- (i) $\|\Phi u\| \leq \|u\|$ for all $u \in P \cap \partial\Omega_1$, and there exists $\psi \in P \setminus \{0\}$ such that $u \neq \Phi u + \lambda\psi$ for all $u \in P \cap \partial\Omega_2$ and $\lambda > 0$, or
- (ii) $\|\Phi u\| \leq \|u\|$ for all $u \in P \cap \partial\Omega_2$, and there exists $\psi \in P \setminus \{0\}$ such that $u \neq \Phi u + \lambda\psi$ for all $u \in P \cap \partial\Omega_1$ and $\lambda > 0$.

Then Φ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In order to obtain a cone in the Banach space X , we define

$$M^* := \max \left\{ q^{hm} \prod_{i=0}^{h-1} a(q^{im}t) : t \in Q_\omega \right\},$$

$$M_* := \min \left\{ q^{hm} \prod_{i=0}^{h-1} a(q^{im}t) : t \in Q_\omega \right\},$$

and

$$\delta := \frac{M_*^2(1 - M^*)}{M^*(1 - M_*)}.$$

Note $0 < \delta < 1$. Now we define the cone P and the mapping $T : X \rightarrow X$ by

$$P := \{ y \in X : y(t) \geq 0, y(t) \geq \delta \|y\|, t \in q^{\mathbb{N}_0} \},$$

$$(Tx)(t) := \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)} \sum_{i=0}^{h-1} \frac{f(q^{im}t, x(q^{im}t/\tau(q^{im}t)))}{\prod_{j=0}^i a(q^{jm}t)},$$

respectively. Since we have

$$\begin{aligned} \frac{q^{hm} M_*}{1 - M_*} \sum_{i=0}^{h-1} f(q^{im}t, x(q^{im}t/\tau(q^{im}t))) &\leq (Tx)(t) \\ &\leq \frac{q^{hm} M^*}{M_*(1 - M^*)} \sum_{i=0}^{h-1} f(q^{im}t, x(q^{im}t/\tau(q^{im}t))) \end{aligned}$$

for any $x \in P$, it follows that $T(P) \subset P$. Define

$$\varphi(s) := \max \left\{ \frac{q^m t f(t, u)}{1 - q^m a(t)} : t \in Q_\omega, \delta s \leq u \leq s \right\},$$

$$\psi(s) := \min \left\{ \frac{q^m \delta f(t, u(t))}{(1 - q^m a(t))u(t)} : t \in Q_\omega, \delta s \leq u \leq s \right\}.$$

Then both functions φ and ψ are continuous on \mathbb{R} .

Theorem 2.2. Assume $0 < a(t) < 1/q^m$ for all $t \in q^{\mathbb{N}_0}$, where m is the order of the functional q -difference (1.1). Suppose there exist two real numbers $\alpha, \beta > 0$ with $\alpha \neq \beta$

such that $\varphi(\alpha) \leq \alpha$ and $\psi(\beta) \geq 1$. Then (1.1) has at least one positive solution $x \in X$ satisfying

$$\min\{\alpha, \beta\} \leq \|x\| \leq \max\{\alpha, \beta\}.$$

Proof. Without loss of generality, we can assume $\alpha < \beta$. Let

$$\Omega_1 := \{x \in X : \|x\| < \alpha\} \quad \text{and} \quad \Omega_2 := \{x \in X : \|x\| < \beta\}.$$

First, we show

$$\|T(x)\| \leq \|x\| \quad \text{for all} \quad x \in P \cap \partial\Omega_1. \quad (2.2)$$

Let $x \in P \cap \partial\Omega_1$. Then $\|x\| = \alpha$ and $\delta\alpha \leq x(t) \leq \alpha$ for all $t \in q^{\mathbb{N}_0}$. Since

$$\frac{q^m t f(t, u)}{1 - q^m a(t)} \leq \varphi(\alpha) \leq \alpha$$

and

$$\frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)} \sum_{i=0}^{h-1} \frac{1 - q^m a(q^{mi}t)}{q^{(i+1)m} \prod_{j=0}^i a(q^{jm}t)} = 1$$

for all $t \in q^{\mathbb{N}_0}$, we obtain

$$\begin{aligned} (Tx)(t) &= \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)} \sum_{i=0}^{h-1} \frac{f(q^{im}t, x(q^{im}t/\tau(q^{im}t)))}{\prod_{j=0}^i a(q^{jm}t)} \\ &\leq \frac{\alpha}{t} \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)} \sum_{i=0}^{h-1} \frac{1 - q^m a(q^{mi}t)}{q^{(i+1)m} \prod_{j=0}^i a(q^{jm}t)} \\ &\leq \alpha = \|x\| \end{aligned}$$

for all $t \in q^{\mathbb{N}_0}$. Hence (2.2) holds. Next, we show that

$$x \neq Tx + \lambda \quad \text{for all} \quad x \in P \cap \partial\Omega_2, \quad \text{for some} \quad \lambda > 0. \quad (2.3)$$

Suppose (2.3) does not hold, i.e., there exist $x^* \in P \cap \partial\Omega_2$ and λ_0 such that $x^* = Tx^* + \lambda_0$.

Let

$$\chi := \min\{x^*(t) : t \in Q_\omega\}.$$

Since $x^* \in P \cap \partial\Omega_2$, $\|x^*\| = \beta$ and $\delta\beta \leq x^*(t) \leq \beta$ for all $t \in q^{\mathbb{N}_0}$. Thus we have $\chi = x^*(t_0)$ for some $t_0 \in Q_\omega$. Since

$$1 \leq \psi(\beta) \leq \frac{q^m \delta f(t_0, u)}{(1 - q^m a(t_0))u}$$

and

$$\frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t_0)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t_0)} \sum_{i=0}^{h-1} \frac{1 - q^m a(q^{im}t_0)}{q^{(1+i)m} \prod_{j=0}^i a(q^{jm}t_0)} = 1,$$

we obtain

$$\begin{aligned} x^*(t_0) &= \lambda_0 + Tx^*(t_0) \\ &= \lambda_0 + \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t_0)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t_0)} \sum_{i=0}^{h-1} \frac{f(q^{im}t_0, x^*(q^{im}t_0/\tau(q^{im}t_0)))}{\prod_{j=0}^i a(q^{jm}t_0)} \\ &\geq \lambda_0 + \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t_0)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t_0)} \sum_{i=0}^{h-1} \frac{(1 - q^m a(q^{im}t_0))x^*(q^{im}t_0/\tau(q^{im}t_0))}{\delta q^m \prod_{j=0}^i a(q^{jm}t_0)} \\ &\geq \lambda_0 + \beta \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t_0)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t_0)} \sum_{i=0}^{h-1} \frac{1 - q^m a(q^{im}t_0)}{q^{(1+i)m} \prod_{j=0}^i a(q^{jm}t_0)} \\ &= \lambda_0 + \beta \\ &\geq \lambda_0 + \chi > \chi. \end{aligned}$$

This gives a contradiction since $x^*(t_0) = \chi$ and hence (2.3) holds. Therefore, by applying Theorem 1.2, it follows that T has a fixed point $x \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This fixed point is a positive ω -periodic solution of (1.1). \square

Corollary 2.3. *Assume $0 < a(t) < 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose that one of the following conditions holds:*

- (i) $\lim_{s \rightarrow 0^+} \frac{\varphi(s)}{s} = \varphi_0 < 1$ and $\lim_{s \rightarrow \infty} \psi(s) = \psi_\infty > 1$,
- (ii) $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = \varphi_\infty < 1$ and $\lim_{s \rightarrow 0^+} \psi(s) = \psi_0 > 1$.

Then (1.1) has at least one positive solution $x \in X$ with $\|x\| > 0$.

Proof. It is sufficient to show only case (i). Since $\lim_{s \rightarrow 0^+} \frac{\varphi(s)}{s} = \varphi_0 < 1$, we choose $\varepsilon = (1 - \varphi_0)/2$ and there exists $\delta > 0$ such that for all $0 < s < \delta$,

$$\frac{3\varphi_0 - 1}{2} < \frac{\varphi(s)}{s} < \frac{1 + \varphi_0}{2} < 1.$$

Then $\alpha \in (0, \delta)$ such that $\varphi(\alpha) < \alpha$. Since $\lim_{s \rightarrow \infty} \psi(s) = \psi_\infty > 1$, we can choose $\varepsilon = (\psi_\infty - 1)/2$ and then we find $\beta > 0$ such that $\psi(\beta) > 1$. Hence, by Theorem 2.2, (1.1) has at least one positive solution $x \in X$ with $\|x\| > 0$. \square

Theorem 2.4. Assume $0 < a(t) < 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose there exist $N + 1$ positive constants $p_1 < p_2 < \dots < p_N < p_{N+1}$ such that one of the following conditions is satisfied:

- (i) $\varphi(p_{2k-1}) < p_{2k-1}$, $k \in \{1, 2, \dots, [(N+2)/2]\}$ and
 $\psi(p_{2k}) > 1$, $k \in \{1, 2, \dots, [(N+1)/2]\}$,
- (ii) $\varphi(p_{2k}) < p_{2k}$, $k \in \{1, 2, \dots, [(N+1)/2]\}$ and
 $\psi(p_{2k-1}) > 1$, $k \in \{1, 2, \dots, [(N+2)/2]\}$,

where $[d]$ denotes the integer part of d . Then (1.1) has at least N positive solutions $x_k \in X$ with

$$p_k < \|x_k\| < p_{k+1} \quad \text{for all } k \in \{1, 2, \dots, N\}.$$

Proof. It is sufficient to show only case (i). Since $\varphi, \psi : (0, \infty) \rightarrow [0, \infty)$ are continuous for each pair $\{p_k, p_{k+1}\}$ and each $k \in \{1, 2, \dots, N\}$, there exist $p_k < \alpha_k < \beta_k < p_{k+1}$ for all $k \in \{1, 2, \dots, N\}$ such that

$$\begin{aligned} \varphi(\alpha_{2k-1}) &< \alpha_{2k-1}, & \psi(\beta_{2k-1}) &> 1, & k &\in \{1, 2, \dots, [(N+2)/2]\}, \\ \varphi(\alpha_{2k}) &< \alpha_{2k}, & \psi(\beta_{2k}) &> 1, & k &\in \{1, 2, \dots, [(N+1)/2]\}. \end{aligned}$$

By Theorem 2.2, (1.1) has at least one positive periodic solution $x_k \in X$ for every pair of numbers $\{\alpha_k, \beta_k\}$ with $p_k < \alpha_k \leq \|x\| \leq \beta_k < p_{k+1}$. The proof is complete. \square

By applying Theorem 2.2, we can easily prove the following two corollaries.

Corollary 2.5. *Assume $0 < a(t) < 1/q^m$ for all $t \in \mathbb{N}_0$. Suppose that the following conditions hold:*

- (i) $\lim_{s \rightarrow 0^+} \frac{\varphi(s)}{s} = \varphi_0 < 1$ and $\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = \varphi_\infty < 1$,
- (ii) *there exists a constant $\beta > 0$ such that $\psi(\beta) > 1$.*

Then (1.1) has at least two positive solutions $x_1, x_2 \in X$ with

$$0 < \|x_1\| < \beta < \|x_2\| < \infty.$$

Corollary 2.6. *Assume $0 < a(t) < 1/q^m$ for all $t \in \mathbb{N}_0$. Suppose that the following conditions hold:*

- (i) $\lim_{s \rightarrow 0^+} \psi(s) = \psi_0 > 1$ and $\lim_{s \rightarrow \infty} \psi(s) = \psi_\infty > 1$,
- (ii) *there exists a constant $\alpha > 0$ such that $\varphi(\alpha) < \alpha$.*

Then (1.1) has at least two positive solutions $x_1, x_2 \in X$ with

$$0 < \|x_1\| < \alpha < \|x_2\| < \infty.$$

3. POSITIVE PERIODIC SOLUTIONS OF (1.2)

In this section, we discuss the existence of positive periodic solutions of (1.2). Throughout this section, we assume $a(t) > \frac{1}{q^m}$ for all $t \in q^{\mathbb{N}_0}$, where m is the order of the functional q -difference equation (1.2). The proofs of the following results are omitted as they can be done similarly to the proofs of the corresponding results in Section 2.

Lemma 3.1. *$x \in X$ is a solution of (1.1) if and only if*

$$x(t) = \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t) - 1} \sum_{i=0}^{h-1} \frac{f(q^{im}t, x(q^{im}t/\tau(q^{im}t)))}{\prod_{j=0}^i a(q^{jm}t)}$$

for all $t \in q^{\mathbb{N}_0}$.

We also define M^* and M_* as in Section 2 but we choose

$$\delta^* := \frac{M_* - 1}{M^*(M^* - 1)}.$$

Clearly, $\delta^* \in (0, 1)$. Then we define the cone

$$P := \{y \in X : y(t) \geq 0, t \in q^{\mathbb{N}_0}, y(t) \geq \delta^* \|y\|\}$$

and the mapping $T : X \rightarrow X$ by

$$Tx(t) = \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t) - 1} \sum_{i=0}^{h-1} \frac{f(q^{im}t, x(q^{im}t/\tau(q^{im}t)))}{\prod_{j=0}^i a(q^{jm}t)}.$$

Thus $Tx(t) = q^\omega Tx(q^\omega t)$ and also $T(P) \subset P$. Define

$$\tilde{\varphi}(s) := \max \left\{ \frac{q^m t f(t, u)}{1 - q^m a(t)} : t \in Q_\omega, \delta^* s \leq u \leq s \right\},$$

$$\tilde{\psi}(s) := \min \left\{ \frac{q^m \delta^* f(t, u(t))}{(1 - q^m a(t))u(t)} : t \in Q_\omega, \delta^* s \leq u \leq s \right\}.$$

Theorem 3.2. Assume $a(t) > 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose there exist two real numbers $\alpha, \beta > 0$ with $\alpha \neq \beta$ such that $\tilde{\varphi}(\alpha) \leq \alpha$ and $\tilde{\psi}(\beta) \geq 1$. Then (1.2) has at least one positive solution $x \in X$ with

$$\min\{\alpha, \beta\} \leq \|x\| \leq \max\{\alpha, \beta\}.$$

Corollary 3.3. Assume $0 < a(t) < 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose that one of the following condition holds:

- (i) $\lim_{s \rightarrow 0^+} \frac{\tilde{\varphi}(s)}{s} = \tilde{\varphi}_0 < 1$ and $\lim_{s \rightarrow \infty} \tilde{\psi}(s) = \tilde{\psi}_\infty > 1$,
- (ii) $\lim_{s \rightarrow \infty} \frac{\tilde{\varphi}(s)}{s} = \tilde{\varphi}_\infty < 1$ and $\lim_{s \rightarrow 0^+} \tilde{\psi}(s) = \tilde{\psi}_0 > 1$.

Then (1.2) has at least one positive solution $x \in X$ with $\|x\| > 0$.

Theorem 3.4. Assume $a(t) > 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose there exist $N + 1$ positive constants $p_1 < p_2 < \dots < p_N < p_{N+1}$ such that one of the following conditions is satisfied:

- (i) $\tilde{\varphi}(p_{2k-1}) < p_{2k-1}$, $k \in \{1, 2, \dots, [(N+2)/2]\}$ and $\tilde{\psi}(p_{2k}) > 1$, $k \in \{1, 2, \dots, [(N+1)/2]\}$,
- (ii) $\tilde{\varphi}(p_{2k}) < p_{2k}$, $k \in \{1, 2, \dots, [(N+1)/2]\}$ and $\tilde{\psi}(p_{2k-1}) > 1$, $k \in \{1, 2, \dots, [(N+2)/2]\}$,

where $[d]$ denotes the integer part of d . Then (1.2) has at least N positive solutions $x_k \in X$, $k \in \{1, 2, \dots, N\}$ with

$$p_k < \|x_k\| < p_{k+1}.$$

Corollary 3.5. Assume $a(t) > 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose that the following conditions are satisfied:

- (i) $\lim_{s \rightarrow 0^+} \frac{\tilde{\varphi}(s)}{s} = \tilde{\varphi}_0 < 1$ and $\lim_{s \rightarrow \infty} \frac{\tilde{\varphi}(s)}{s} = \tilde{\varphi}_\infty < 1$,
- (ii) there exists a constant $\beta > 0$ such that $\tilde{\psi}(\beta) > 1$.

Then (1.2) has at least two positive solutions $x_1, x_2 \in X$ with

$$0 < \|x_1\| < \beta < \|x_2\| < \infty.$$

Corollary 3.6. Assume $a(t) > 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose the following conditions are satisfied:

- (i) $\lim_{s \rightarrow 0^+} \tilde{\psi}(s) = \tilde{\psi}_0 > 1$ and $\lim_{s \rightarrow \infty} \tilde{\psi}(s) = \tilde{\psi}_\infty > 1$,
- (ii) there exists a constant $\alpha > 0$ such that $\tilde{\varphi}(\alpha) < \alpha$.

Then (1.2) has at least two positive solutions $x_1, x_2 \in X$ with

$$0 < \|x_1\| < \alpha < \|x_2\| < \infty.$$

4. SOME EXAMPLES

In this section, we show some examples of equations of the form (1.1) and (1.2) and apply the main results of the previous sections.

Example 4.1. Consider the q -difference equation

$$x(q^3t) = ax(t) + \frac{1}{tx(q^2t)}, \quad (4.1)$$

where a is a constant with $0 < a < 1/q^3$, $f(t, x) = 1/(tx)$, and $\tau(t) = 1/q^2$ for all $t \in q^{\mathbb{N}_0}$.

We have

$$\lim_{s \rightarrow \infty} \frac{\varphi(s)}{s} = \varphi_\infty = 0 < 1 \quad \text{and} \quad \lim_{s \rightarrow 0^+} \psi(s) = \psi_0 = \infty > 1.$$

By Corollary 2.3 (ii), (4.1) has at least one positive ω -periodic solution.

Example 4.2. Let $q = 2$, $m = 4$, $\omega = 5$. Consider the q -difference equation

$$x(16t) = ax(t) + t^{99}x^{100}(4t) + \frac{1}{16000te^{tx(4t)}}, \quad (4.2)$$

where a is a constant with $0 < a < 1/20$, $f(t, x) = t^{99}x^{100} + 1/(16000te^{tx})$, and $\tau(t) = 1/4$ for all $t \in q^{\mathbb{N}_0}$. We have

$$\lim_{s \rightarrow \infty} \psi(s) = \psi_\infty = \infty > 1 \quad \text{and} \quad \lim_{s \rightarrow 0^+} \psi(s) = \psi_0 = \infty > 1.$$

Since there exists $\alpha = 1/100$ such that $\varphi(\alpha) < \alpha$, by Corollary 2.6, (4.2) has at least two positive ω -periodic solutions.

Example 4.3. Consider the q -difference equation

$$x(q^5t) = a_t x(t) - t^2 x^3(qt), \quad (4.3)$$

where $a(t) = a_t$ are constants assigned for each $t \in Q_\omega$ and $a(t) = a(q^\omega t)$ for all $t \in q^{\mathbb{N}_0}$.

We have $\tau(t) = 1/q$, $f(t, x) = t^2 x^3$,

$$\lim_{s \rightarrow 0^+} \frac{\tilde{\varphi}(s)}{s} = \tilde{\varphi}_0 = 0 < 1 \quad \text{and} \quad \lim_{s \rightarrow \infty} \tilde{\psi}(s) = \tilde{\psi}_\infty = \infty > 1.$$

By Corollary 3.3 (i), (4.3) has at least one positive ω -periodic solution.

5. REFERENCES

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SECTION

4. CONCLUSION

We now summarize and comment on the new results and approaches presented in our study of periodic solutions of q -difference equations.

In the our first paper, *Floquet theory for q -difference equations*, the basic Floquet theory is derived on the q -time scale, in analogy with existing theories for the time scales \mathbb{Z} and \mathbb{R} , for the Floquet equation, $x^\Delta = A(t)x$, where A is assumed to be regressive and ω -periodic. The regressive property of A is seen to be necessary in the q -time scale setting and the definition of periodicity for functions on a q -time scale is based on integration in distinction with the standard approach taken for \mathbb{Z} and \mathbb{R} . The representation of the fundamental matrix of the q -Floquet equation is presented in Theorem 4.2, and results analogous to those which exist for \mathbb{Z} and \mathbb{R} are presented for the Floquet equation for the q -time scale setting in Theorems 4.3 and 4.7.

The stability of solutions for Floquet equations in the q -time scale setting will be considered in future works. For the present, we briefly sketch some issues that arise in connection with this study: Suppose the q -Floquet equation, with some initial conditions given, has n solutions and they are represented by an infinite sequence in $t \in q^{\mathbb{N}}$ of points $(u_1(t), \dots, u_n(t))$ in \mathbb{R}^n . In many applications of this subject, it is useful to know the general location of those points for the large values of time t . Central to this study is the consideration and analysis of several possibilities that arise: the sequence may converge to a point or at least remain near a point; the sequence may oscillate among values near several points; the sequence may become unbounded; or the sequence may remain in a bounded set but jump around in a seemingly unpredictable fashion.

In our second paper, *The Beverton–Holt q -difference equation*, we consider

$$x^\Delta(t) = a(t)x^\sigma(t) \left(1 - \frac{x(t)}{K(t)} \right),$$

where $a(t) = \frac{\alpha}{t}$ and $K(t) = q^\omega K(q^\omega t)$ for all $t \in \mathbb{T} = q^{\mathbb{N}_0}$, α a constant. Given that $a(t) = \frac{\alpha}{t}$ is 1-periodic, it follows from our definition of periodicity on the q -time scale that the function a is also ω -periodic for any $\omega > 1$. We have derived the periodic solutions

of our Beverton–Holt q -difference equation and, as in the \mathbb{Z} setting, the Cushing–Henson conjectures for our Beverton–Holt q -difference equation have been presented. Other close forms of the function a which generalize the Beverton–Holt q -difference equation remain to be studied.

In our third paper, *Stability for Hamiltonian q -difference systems*, we have derived the stability theory for Hamiltonian q -difference systems. Our work on locating zones of stability for the Hamiltonian q -difference systems is based on the work of Krein and Jakubovič [22], and Răsvan [17, 27].

For Hamiltonian q -difference systems without the parameter, multipliers which have modulus one and are of simple type indicate that the solutions of the Hamiltonian q -difference system are bounded; in other words, the Hamiltonian q -difference system is weakly stable. Furthermore, a sufficient condition for strong stability of the Hamiltonian q -difference system is that all its multipliers lie on the unit circle and are definite.

A Hamiltonian q -difference system with parameter is stable if the monodromy matrix is of stable type, i.e., all multipliers are simple and have modulus one which may be the first kind, second kind, or mixed kind. If all multipliers are simple with multiplicity one, and the stability is strong for a sufficiently small perturbation, the multipliers cannot leave the unit circle since they will break up the symmetry of multipliers. Multipliers possessing multiplicity of at least two, may be located away from the unit circle. A meeting of multipliers of the same kind will not move away from the unit circle, while multipliers of different kinds that meet on the unit circle may move off the unit circle under a suitable perturbation. Given our definition of periodicity on the q -time scale, our results in this paper are slightly different from their analogs in the \mathbb{Z} and \mathbb{R} settings.

In our fourth paper, *Existence of periodic solutions of a q -difference boundary value problem*, we consider the second order q -difference BVP,

$$\begin{aligned} x^{\Delta\Delta}(t) + \nabla F(qt, x(qt)) &= 0, \\ x(1) &= 1/q^{\omega/2}x(q^\omega), \quad x^\Delta(1) = q^{\omega/2}x^\Delta(q^\omega), \end{aligned}$$

where $F(t, u)$ is continuously differentiable in u and ω -periodic in t . Existence theorems for solutions of our second order q -difference BVP subject to specific boundary conditions have been proven by applying the Mountain Pass Theorem. In this context, dependence on F is characterized. Explicit representations for periodic solutions associated with given boundary conditions remains a topic to be explored. However, we have found that the reciprocal square of the solutions of the second order q -difference are periodic where, in general, they are not the solutions of our second order q -difference BVP.

In our fifth and final paper, *Positive periodic solutions of higher-order functional q -difference equations*, we consider higher-order functional q -difference equations of the form,

$$x(q^m t) = a(t)x(t) + f(t, x(t/\tau(t))).$$

We have obtained existence theorems for solutions of two higher-order functional q -difference equations and found the closed forms of those solutions. In studying positive solutions for these equations the following conditions on a were found to be significant: $0 < a(t) < 1/q^m$ or $a(t) > 1/q^m$ must be held for all $t \in q^{\mathbb{N}_0}$, where m is the order of that equations. Under these conditions, $a(t)$ will get small or large depending on the order, m , of the equation considered. In considering the high order of functional q -difference equations, one must deal with very small or very large values of the function a which may yield difficulties in numerical calculations; a topic for future exploration.

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